On the Darboux and Birkhoff steps in the asymptotic stability of solitons

Scipio Cuccagna

February 25, 2013

Abstract

We give a unified proof of the step to find Darboux coordinates and of the ensuing Birkhoff normal forms procedure, developed in the course of the proof of asymptotic stability of solitary waves in [4, 6, 7].

1 Introduction

The aim of this paper is to extend in a slightly more general and unified set up two important steps of the proof of the asymptotic stability of solitary waves for the Nonlinear Schrödinger equation [7, 6, 2] and the particular case of Nonlinear Dirac system treated in [4]. In both cases there is a localization at the solitary wave and a representation of the system in terms of coordinates arising from the linearization at a solitary wave. The operators $\mathcal{H}_p$ introduced later play this role. In general $\mathcal{H}_p$ has both continuous spectrum and non zero eigenvalues. The latter give rise to discrete modes which in the nonlinear problem could produce chaotic Lissaius like motions. It turns out that in [3, 7, 9, 4, 6, 2] discrete modes relax to 0 because of a mechanism of slow leaking of energy away from the discrete modes into the continuous modes, where energy disperses by linear dispersion. The idea was initiated in special situations in [11, 5, 12]. We refer to [7] for more comments and references.

The aim of this paper consists in simplifying two key steps in the proofs in [6, 7, 4]. The first step consists in searching Darboux coordinates. This allows to decrease the number of coordinates in the system and to reduce to the study of the system at an equilibrium point.

The second step consists in the implementation of the Birkhoff normal forms, to produce a simple *effective* Hamiltonian. After this, [6, 7, 4] prove the energy leaking away from the discrete modes. In particular the key step is the proof that certain coefficients of the discrete modes equations are second powers, the *Nonlinear Fermi Golden Rule* (FGR), which generically are positive and yield discrete mode energy dissipation.

We do not discuss the FGR in this paper limiting ourselves to the search of Darboux coordinates and to the Birkhoff normal forms argument.

In this paper we avail ourselves with some ideas and notation drawn from early versions of [2] to improve the presentation in [6].

[2, 6] represent two attempts to extend the result proved in [7] for standing ground states of the NLS, to the case of moving ground states. A further goal in [2] is to develop the theory in a more abstract set up. Early versions of [2] did not encompass a Birkhoff step extendable to [4]. [2] is confined (like us here) to systems with Abelian group of symmetries.

Our present proof was written before the 3rd version of [2] was posted on the Arxiv site. The 2rd version of [2] contained an incorrect effective Hamiltonian, see Remark 6.7 later. In the 3rd version of [2] this has been corrected. Still, the discussion in [2] is at times sketchy, for example in Theorems 3.21 and 5.2 [2], see Remarks 2.10 and 6.6 and the discussion below and at the beginning of Sect. 3.3 about (3.42).

We draw from [2] a better choice of initial coordinates and set up than [6]. Some of it existed also in previous literature, cfr. the discussion in Sect.6 [10]. We also borrow some notation, i.e. symbols $\mathcal{R}^{k,m}$ and $\mathbf{S}^{k,m}$. Inspired by [2] we simplify the proof of the Birkhoff step in [6].

Both here and in [6] we consider initial data in subsets of Σ_n for $n \gg 1$ which are unbounded in Σ_n and invariant for the system. We require this substantial amount of regularity and spacial decay to 0 for the classes of solutions of the system, in order to give a rigorous treatment of the flows and of the pullbacks. [2] suggests that [6] should prove decay rates in time. We do not know what is the basis for this statement in [2] since, by the time invariance of the subsets Σ_n considered, the problem considered in [6] is very similar in this respect to the one with Σ_n replaced by H^1. Indeed time decay corresponds to bounds on norms containing time dependent weights. But if the problem is invariant by translation in time, the only information that can be derived must be invariant by translation in time, and bounds on time weighted norms do not have this property. We therefore emphasize that [6] and the present paper are very different from, say, [5, 12], which consider initial data in subsets of $H^{k,s}$ which are not invariant by the time evolution. We also point out that the 1st version of [2] contains a false statement on rate of time decay. The 2nd version of [2] in the acknowledgments, credits us for pointing out this error, although these credits are not any more in the 3rd version.

To find an effective Hamiltonian, we use in a crucial way the regularity properties of the flows, which in turn depend on the fact that we work in Σ_n for $n \gg 1$. See Theorem 6.5 where the regularity of the flows is used to prove that the coordinate changes preserve the system. To prove for the NLS the same result in H^1, where the coordinate changes are continuous only, one needs to explain how they preserve the structure needed to make sense of the NLS. [2] claims the result in H^1, [2] claims the result in H^1, but the proof is not spelled out, see Remark 6.6.

We discuss in some detail a key formula on the differentiation of the pullback of a differential form along a flow, see (3.42), which is the basis of Moser's method to find Darboux coordinates. This formula is simple in classical set ups, but in our case and in [2] its interpretation and proof are not obvious.

In [2] the formula is stated and used without discussion. We treat the issue rigorously in Sect. 3.3, regularizing the flow, using (3.42) for the regularized flow, and recovering the desired equality between differential forms, by a limiting argument. Notice that we do not prove formula (3.42) for the non regularized flow.

We end with few remarks on the proofs.

The proof of the Darboux Theorem is a simplification of that in [6] in the part discussing the vector field. We give in Sect. 3.3 a detailed proof on the fact that the resulting flow transforms the symplectic form as desired. See also the introductory remarks in Sect. 3. Notice that parts of this discussion were skipped in [6].

The portion of our paper on the Birkhoff normal forms covers from Sect. 4 on and is quite different from [4, 6, 7] mainly because the pullback of the terms of the expansion of the Hamiltonian cannot be treated on a term by term basis, see Remark 5.5. What is important is to get a general structure of the pullbacks of the Hamiltonian. This is discussed in Sect. 4. It is likely that most of the analysis in Lemmas 4.3, 4.4 and 5.4, is not necessary to the derivation of the effective Hamiltonian, which is represented by H_2' and the null terms in $\mathbf{R}_0$ and $\mathbf{R}_1$ of the expansion in Lemma 5.4, in the final Hamiltonian. On the other hand, writing the Hamiltonian explicitly should make the arguments transparent and more clearly applicable to the part on dispersion and Fermi Golden rule.

In Sect. 5 we finally distinguish between discrete and continuous modes.

The present paper treats only equations whose symmetry group is Abelian. This limitation will have to be overcome to extend the theory to more general systems such for example the Dirac system without the symmetry constraints of [4].

2 Set up

- Given two vectors $u, v \in \mathbb{R}^{2N}$ we denote by $u \cdot v = \sum u_j v_j$ their inner product.
- We will consider also another quadratic form $|u|_1^2 = u \cdot_1 u$ in $\mathbb{R}^{2N}$.
- For any $n \geq 1$ we consider the space $\Sigma_n = \Sigma_n(\mathbb{R}^3, \mathbb{R}^{2N})$ defined by

  $$\|u\|_{\Sigma_n}^2 := \sum_{|\alpha| \leq n} (\|x^\alpha u\|_{L^2(\mathbb{R}^3, \mathbb{R}^{2N})}^2 + \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^3, \mathbb{R}^{2N})}^2) < \infty.$$

  We set $\Sigma_0 = L^2(\mathbb{R}^3, \mathbb{R}^{2N})$. Equivalently we can define Σ_r for $r \in \mathbb{R}$ by the norm

  $$\|u\|_{\Sigma_r} := \|(1 - \Delta + |x|^2)^{\frac{r}{2}} u\|_{L^2} < \infty.$$

  For $r \in \mathbb{N}$ the two definitions are equivalent, see [7]. We will not use another quite natural class of spaces denoted by $H^{k,s}$ and defined by

  $$\|u\|_{H^{k,s}} := \|(1 + |x|^2)^{\frac{s}{2}} (1 - \Delta)^{\frac{k}{2}} u\|_{L^2} < \infty.$$

- $\mathcal{S}(\mathbb{R}^3,\mathbb{R}^{2N}) = \cap_{n\in\mathbb{N}}\Sigma_n(\mathbb{R}^3,\mathbb{R}^{2N})$ is the space of Schwartz functions and the space of tempered distributions is $\mathcal{S}'(\mathbb{R}^3,\mathbb{R}^{2N}) = \cup_{n\in\mathbb{N}}\Sigma_{-n}(\mathbb{R}^3,\mathbb{R}^{2N})$.

- For X and Y two Banach space, we will denote by $B(X,Y)$ the Banach space of bounded linear operators from X to Y and by $B^\ell(X,Y) = B(\prod_{j=1}^{\ell} X, Y)$.

- We denote by $\langle\ ,\ \rangle$ the natural inner product in $L^2(\mathbb{R}^3,\mathbb{R}^{2N})$.

- J is an invertible antisymmetric matrix in $\mathbb{R}^{2N}$. We have also $|Jy|_1 = |y|_1$ for any $y\in\mathbb{R}^{2N}$. In $L^2(\mathbb{R}^3,\mathbb{R}^{2N})$ we consider the symplectic form $\Omega = \langle J^{-1}\ ,\ \rangle$.

- We consider in $L^2(\mathbb{R}^3,\mathbb{R}^{2N})$ a linear selfadjoint elliptic differential operator $\mathcal{D}$ such that $\mathcal{D}\in B(\Sigma_r,\Sigma_{r-\mathrm{ord}\mathcal{D}})$ and $\mathcal{D}\in B(H^r,H^{r-\mathrm{ord}\mathcal{D}})$ for all r and for a fixed integer $\mathrm{ord}\mathcal{D}\geq 1$.

- We consider a Hamiltonian of the form

$$\begin{aligned} &E(U) = E_K(U) + E_P(U) \\ &E_K(U) := \frac{1}{2}\langle\mathcal{D}U,U\rangle\,, \quad E_P(U) := \int_{\mathbb{R}^3} B(|U|_1^2)dx. \end{aligned} \tag{2.1}$$

  Here $B\in C^\infty(\mathbb{R},\mathbb{R})$, $B(0)=B'(0)=0$ and there exists a $p\in(2,6]$ such that for every $k\geq 0$ there is a fixed C_k with

$$\left|\nabla_\zeta^k(B(|\zeta|_1^2))\right| \leq C_k|\zeta|^{p-k-1} \quad \text{if } |\zeta|\geq 1 \text{ in } \mathbb{R}^{2N}. \tag{2.2}$$

Notice that $E_P\in C^5(H^1(\mathbb{R}^3,\mathbb{R}^{2N})),\mathbb{R})$. Consistently with [4, 6, 7], we focus only on *semilinear* Hamiltonians. We consider the system

$$\dot{U} = J\nabla E(U) \quad , \quad U(0) = U_0 \tag{2.3}$$

where for a Frechét differentiable function F the gradient $\nabla F(U)$ is defined by $\langle\nabla F(U),X\rangle = dF(U)(X)$, with $dF(U)$ the exterior differential calculated at U. We assume that

(A1) there exists d_0 such that for $d>d_0$ system (2.3) is locally well posed in H^d. Furthermore, the space Σ_d is invariant by this motion.

We recall the following definition.

Definition 2.1. Given a Frechét differentiable function F, the Hamiltonian vectorfield of F with respect to a *strong* symplectic form ω, see [1] Ch. 9, is the field X_F such that $\omega(X_F,Y) = dF(Y)$ for any given tangent vector Y. For $\omega=\Omega$ we have $X_F = J\nabla F$.

For F,G two scalar Frechét differentiable functions, we consider the Poisson bracket $\{F,G\} := dF(X_G)$.

If $\mathcal{G}$ has values in a given Banach space $\mathbb{E}$ and G is a scalar valued function, then we set $\{\mathcal{G},G\} := \mathcal{G}'(X_G)$, for $\mathcal{G}'$ the Frechét derivative of $\mathcal{G}$.

We assume some symmetries in system (2.3). Specifically we assume what follows.

(A2) There are selfadjoint differential operators $\Diamond_\ell$ for $\ell = 1, ..., n_0$ in L^2 such that $\Diamond_\ell : \Sigma_n \to \Sigma_{n-d_\ell}$ for $\ell = 1, ..., n_0$. We set $\mathbf{d} = \sup_\ell d_\ell$.

(A3) We assume $[\Diamond_\ell, J] = 0$ and $[\Diamond_\ell, \Diamond_k] = 0$.

(A4) We assume $\{\Pi_\ell, E_K\} = \{\Pi_\ell, E_P\} = 0$ for all ℓ, where $\Pi_\ell := \frac{1}{2}\langle \Diamond_\ell \ , \ \rangle$.

(A5) Set $\langle \epsilon \Diamond \rangle^2 := 1 + \sum_j \epsilon^2 \Diamond_j^2$. Then $\langle \epsilon \Diamond \rangle^{-2} \in B(\Sigma_n, \Sigma_n)$ with

$$\| \langle \epsilon \Diamond \rangle^{-2} \|_{B(\Sigma_n, \Sigma_n)} \le C_n < \infty \text{ for any } |\epsilon| \le 1 \text{ and } n \in \mathbb{N}. \tag{2.4}$$

Furthermore, for any $n \in \mathbb{Z}$ we have

$$\begin{aligned} & strong - \lim_{\epsilon \to 0} \langle \epsilon \Diamond \rangle^{-2} = 1 \text{ in } B(\Sigma_n, \Sigma_n) \\ & \lim_{\epsilon \to 0} \| \langle \epsilon \Diamond \rangle^{-2} - 1 \|_{B(\Sigma_n, \Sigma_{n'})} = 0 \quad \text{ for any } n' \in \mathbb{Z} \text{ with } n' < n. \end{aligned} \tag{2.5}$$

(A6) Consider the groups $e^{J \langle \epsilon \Diamond \rangle^{-2} \Diamond \cdot \tau}$ defined in L^2. We assume that for any $n \in \mathbb{N}$ these groups leave Σ_n invariant and that for any $n \in \mathbb{N}$ and $c > 0$ there a C s.t. $\| e^{J \langle \epsilon \Diamond \rangle^{-2} \Diamond \cdot \tau} \|_{B(\Sigma_n, \Sigma_n)} \le C$ for any $|\tau| \le c$ and any $|\epsilon| \le 1$.

We introduce now our *solitary* waves.

(B1) We assume that for $\mathcal{O}$ an open subset of $\mathbb{R}^{n_0}$ we have a function $p \to \Phi_p \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}^{2N})$ which is in $C^\infty(\mathcal{O}, \mathcal{S})$, with $\Pi_\ell(\Phi_p) = p_\ell$, where the Φ_p are constrained critical points of E with associated Lagrange multipliers $\lambda_\ell(p)$ so that

$$\nabla E(\Phi_p) = \lambda(p) \cdot \Diamond \Phi_p \tag{2.6}$$

(B2) We will assume that the map $p \to \lambda(p)$ is a diffeomorphism. In particular this means that the following matrix has rank n_0

$$\text{rank} \left[\frac{\partial \lambda_i}{\partial p_j} \right]_{i\downarrow \ , \ j\rightarrow} = n_0. \tag{2.7}$$

A function $U(t) := e^{J(t\lambda(p) + \tau_0)\cdot \Diamond} \Phi_p$ is a solitary wave solution of (2.3) for any fixed vector τ_0.

2.1 The linearization

Set $\mathcal{H}_p := J(\nabla^2 E(\Phi_p) - \lambda(p)\cdot\Diamond)$. Notice that $E(e^{J\tau\cdot\Diamond}U) \equiv E(U)$ for any U yields $\nabla E(e^{J\tau\cdot\Diamond}U) = e^{J\tau\cdot\Diamond}\nabla E(U)$ and $\nabla^2 E(e^{J\tau\cdot\Diamond}U) = e^{J\tau\cdot\Diamond}\nabla^2 E(U)e^{-J\tau\cdot\Diamond}$. Then (2.6) implies $\nabla E(e^{J\tau\cdot\Diamond}\Phi_p) = e^{J\tau\cdot\Diamond}\lambda(p)\cdot\Diamond\Phi_p$. So applying ∂_{τ_j} we obtain $(\nabla^2 E(\Phi_p) - \lambda(p)\cdot\Diamond)J\Diamond_j\Phi_p = 0$ and so

$$\mathcal{H}_p J\Diamond_j\Phi_p = 0 \tag{2.8}$$

(C1) We will assume

$$\ker\mathcal{H}_p = \mathrm{Span}\{J\Diamond_j\Phi_p : j = 1, ..., n_0\}. \tag{2.9}$$

Applying ∂_{λ_j} to (2.6) yields $(\nabla^2 E(\Phi_p) - \lambda(p)\cdot\Diamond)\partial_{\lambda_j}\Phi_p = \Diamond_j\Phi_p$. This yields

$$\mathcal{H}_p\partial_{\lambda_j}\Phi_p = J\Diamond_j\Phi_p \tag{2.10}$$

We have

$$\langle\partial_{\lambda_j}\Phi_p, \Diamond_k\Phi_p\rangle = \frac{1}{2}\partial_{\lambda_j}\langle\Phi_p, \Diamond_k\Phi_p\rangle = \partial_{\lambda_j}p_k. \tag{2.11}$$

Necessarily, by (B2) there exists j such that $\partial_{\lambda_j}p_k \neq 0$. This implies that the *generalized kernel* is

$$N_g(\mathcal{H}_p) = \mathrm{Span}\{J\Diamond_j\Phi_p, \partial_{\lambda_j}\Phi_p : j = 1, ..., n_0\}. \tag{2.12}$$

The map $(p,\tau) \to e^{J\tau_0\cdot\Diamond}\Phi_p$ is in $C^\infty(\mathcal{O}\times\mathbb{R}^{n_0}, \mathcal{S})$.

(C2) We assume this map is a local embedding and that the image is a manifold $\mathcal{G}$.

At any given point $e^{J\tau\cdot\Diamond}\Phi_p$ the tangent space of $\mathcal{G}$ is given by

$$T_{e^{J\tau\cdot\Diamond}\Phi_p}\mathcal{G} = \mathrm{Span}\{e^{J\tau\cdot\Diamond}\partial_{p_j}\Phi_p, e^{J\tau\cdot\Diamond}\Diamond_j\Phi_p : j = 1, ..., n_0\}.$$

We have $\Omega(e^{J\tau\cdot\Diamond}\partial_{p_j}\Phi_p, e^{J\tau\cdot\Diamond}\partial_{p_k}\Phi_p) = \Omega(\partial_{p_j}\Phi_p, \partial_{p_k}\Phi_p)$.

(C3) We assume that

$$\Omega(\partial_{p_j}\Phi_p, \partial_{p_k}\Phi_p) = 0 \text{ for all } j \text{ and } k \tag{2.13}$$

$$\Omega(\partial_{p_j}\Phi_p, \Phi_p) = 0 \text{ for all } j. \tag{2.14}$$

Notice that (2.14) is not required in [2] but in any case is true for the applications in [2, 4, 6, 7]. Here we use it in Lemma 3.1.

We have the following beginning of Jordan block decomposition of $\mathcal{H}_p$.

Lemma 2.2. *Consider the operator $\mathcal{H}_p$. We have*

$$J^{-1}\mathcal{H}_p = -\mathcal{H}_p^* J^{-1}, \quad \mathcal{H}_p J = -J\mathcal{H}_p^*. \tag{2.15}$$

Assume (B1)–(B2) and (C1). Then we have

$$L^2 = N_g(\mathcal{H}_p) \oplus N_g^{\perp}(\mathcal{H}_p^*) \ , \tag{2.16}$$

$$N_g(\mathcal{H}_p^*) = Span\{\Diamond_j \Phi_p, J^{-1}\partial_{\lambda_j}\Phi_p : j = 1, ..., n_0\}. \tag{2.17}$$

Proof. We have $\mathcal{H}_p = JA$ for a selfadjoint operator A and with J a bounded antisymmetric operator. Then $\mathcal{H}_p^* = -AJ$ and (2.15) follows by direct inspection. Recall that (B1)–(B2) and (C1) imply (2.12). Then (2.15) implies (2.17).
The map $\psi \to \langle \ , \psi \rangle$ establishes a map $N_g(\mathcal{H}_p^*) \to B(N_g(\mathcal{H}_p), \mathbb{R})$. By (2.11), formulas (2.12) and (2.17) imply that this map is an isomorphism. For any $u \in L^2$ there is exactly one $v \in N_g(\mathcal{H}_p)$ such that $\langle u, \ \rangle$ and $\langle v, \ \rangle$ coincide as elements in $B(N_g(\mathcal{H}_p^*), \mathbb{R})$. Then $u - v \in N_g^{\perp}(\mathcal{H}_p^*)$ and we get (2.16).

$\square$

Obviously Lemma 2.2 holds true only because our J is very special. For the KdV, where $J = \frac{\partial}{\partial x}$, (2.16)–(2.17) are not true.
Denote by $P_{N_g}(p) = P_{N_g(\mathcal{H}_p)}$ the projection onto $N_g(\mathcal{H}_p)$ associated to (2.16) and by $P(p) := 1 - P_{N_g}(p)$ the projection on $N_g^{\perp}(\mathcal{H}_p^*)$. We have, summing on repeated indexes,

$$P_{N_g}(p)X = -J\Diamond_j\Phi_p \ \langle X, J^{-1}\partial_{p_j}\Phi_p\rangle + \partial_{p_j}\Phi_p \ \langle X, \Diamond_j\Phi_p\rangle. \tag{2.18}$$

Lemma 2.3. *Assume (B1)–B(2) and (C1). Then:*

(1) $P_{N_g}(p) \in B(\mathcal{S}', \mathcal{S})$ *for any* $p \in \mathcal{O}$ *and* $P_{N_g}(p) \in C^{\infty}(\mathcal{O}, B(\Sigma_{-k}, \Sigma_k))$ *for any* $k \in \mathbb{N}$.

(2) For any $p_0 \in \mathcal{O}$ *and* k *there exists an* $\varepsilon_k > 0$ *such that for* $|p - p_0| < \varepsilon_k$

$$P(p)P(p_0) : N_g^{\perp}(\mathcal{H}_{p_0}^*) \cap \Sigma_k \to N_g^{\perp}(\mathcal{H}_p^*) \cap \Sigma_k \tag{2.19}$$

is an isomorphism.

(3) For $h > k$ *we have* $\varepsilon_h \geq \varepsilon_k$.

Proof. Claim (1) is elementary and we skip the proof.
Consider the map $P(p)P(p_0)P(p) = 1 + P(p)(P_{N_g}(p) - P_{N_g}(p_0))P(p)$ from $N_g^{\perp}(\mathcal{H}_p^*) \cap \Sigma_k$ into itself. By Claim (1) and by the Fredholm alternative, this is an isomorphism for $|p - p_0| < \varepsilon_k$ with $\varepsilon_k > 0$ sufficiently small. This implies that the $P(p)P(p_0)$ in (2.19) is onto. For the same reasons also $P(p_0)P(p)P(p_0)$ is an isomorphism from $N_g^{\perp}(\mathcal{H}_{p_0}^*) \cap \Sigma_k$ into itself. Then $P(p)P(p_0)$ in (2.19) is one to one. This yields Claim (2).
For $h > k$ we have the commutative diagram

$$\begin{array}{ccc} N_g^{\perp}(\mathcal{H}_{p_0}^*) \cap \Sigma_h & \overset{P(p)P(p_0)}{\to} & N_g^{\perp}(\mathcal{H}_p^*) \cap \Sigma_h \\ \downarrow & & \downarrow \\ N_g^{\perp}(\mathcal{H}_{p_0}^*) \cap \Sigma_k & \overset{P(p)P(p_0)}{\to} & N_g^{\perp}(\mathcal{H}_p^*) \cap \Sigma_k \end{array}$$

with the vertical maps two embedding. This implies that for $|p-p_0|<\varepsilon_k$ we have $\ker P(p)P(p_0)=0$ in $N_g^{\perp}(\mathcal{H}^*_{p_0})\cap\Sigma_h$. To complete the proof of Claim (3), we need to show that given $u\in N_g^{\perp}(\mathcal{H}^*_p)\cap\Sigma_h$ and the resulting $v\in N_g^{\perp}(\mathcal{H}^*_{p_0})\cap\Sigma_k$ with $u=P(p)P(p_0)v$, we have $v\in\Sigma_h$. But this follows immediately from

$$v=u+(P_{N_g}(p)-P_{N_g}(p_0))v \text{ where } u\in\Sigma_h \text{ and } (P_{N_g}(p)-P_{N_g}(p_0))v\in\mathcal{S}.$$

□

We will denote the inverse of (2.19) by

$$(P(p)P(p_0))^{-1}:N_g^{\perp}(\mathcal{H}^*_p)\cap\Sigma_k\to N_g^{\perp}(\mathcal{H}^*_{p_0})\cap\Sigma_k. \tag{2.20}$$

We have the following *Modulation* type lemma.

Lemma 2.4 (Modulation)**.** *Assume (A2), (B.1), (B.2), (C.1) and (C.3). Fix $n\in\mathbb{Z}$, $n\geq 0$ and fix $\Psi_0=e^{J\tau_0\cdot\lozenge}\Phi_{p_0}$. Then $\exists$ a neighborhood $\mathcal{U}$ in $\Sigma_{-n}(\mathbb{R}^3,\mathbb{R}^{2N})$ of U_0 and functions $p\in C^\infty(\mathcal{U},\mathcal{O})$ and $\tau\in C^\infty(\mathcal{U},\mathbb{R}^{n_0})$ s.t. $p(\Psi_0)=p_0$ and $\tau(\Psi_0)=\tau_0$ and s.t. $\forall U\in\mathcal{U}$*

$$U=e^{J\tau\cdot\lozenge}(\Phi_p+R) \text{ and } R\in N_g^{\perp}(\mathcal{H}^*_p). \tag{2.21}$$

Proof. Consider the following $2n_0$ functions:

$$\begin{aligned}\mathcal{F}_j(U,p,\tau)&:=\Omega(U-e^{J\tau\cdot\lozenge}\Phi_p,e^{J\tau\cdot\lozenge}\partial_{p_j}\Phi_p)\\ \mathcal{G}_j(U,p,\tau)&:=\Omega(U-e^{J\tau\cdot\lozenge}\Phi_p,Je^{J\tau\cdot\lozenge}\lozenge_j\Phi_p).\end{aligned} \tag{2.22}$$

These functions belong to $C^\infty(\Sigma_{-n}\times\mathcal{O}\times\mathbb{R}^{n_0},\mathbb{R})$. We introduce the notation $R=e^{-J\tau\cdot\lozenge}U-\Phi_p$. Notice that $R=0$ for $U=\Phi_p$. Then

$$\begin{aligned}&\partial_{\tau_k}\mathcal{F}_j(U,p,\tau)=\Omega(e^{J\tau\cdot\lozenge}R,e^{J\tau\cdot\lozenge}J\lozenge_k\partial_{p_j}\Phi_p)-\Omega(J\lozenge_k e^{J\tau\cdot\lozenge}\Phi_p,e^{J\tau\cdot\lozenge}\partial_{p_j}\Phi_p)\\ &=-\langle R,\lozenge_k\partial_{p_j}\Phi_p\rangle-\langle\lozenge_k\Phi_p,\partial_{p_j}\Phi_p\rangle=-\langle R,\lozenge_k\partial_{p_j}\Phi_p\rangle-\frac{1}{2}\partial_{p_j}\langle\lozenge_k\Phi_p,\Phi_p\rangle\\ &=-\langle R,\lozenge_k\partial_{p_j}\Phi_p\rangle-\delta_{jk}.\end{aligned}$$

By (2.13) we have

$$\begin{aligned}&\partial_{p_k}\mathcal{F}_j(U,p,\tau)=\Omega(e^{J\tau\cdot\lozenge}R,e^{J\tau\cdot\lozenge}\partial_{p_k}\partial_{p_j}\Phi_p)-\Omega(Je^{J\tau\cdot\lozenge}\partial_{p_k}\Phi_p,e^{J\tau\cdot\lozenge}\partial_{p_j}\Phi_p)\\ &=\Omega(R,\partial_{p_k}\partial_{p_j}\Phi_p).\end{aligned}$$

By (A3) we have

$$\begin{aligned}\partial_{\tau_k}\mathcal{G}_j&=\Omega(e^{J\tau\cdot\lozenge}R,e^{J\tau\cdot\lozenge}J^2\lozenge_k\lozenge_j\Phi_p)-\Omega(J\lozenge_k e^{J\tau\cdot\lozenge}\Phi_p e^{J\tau\cdot\lozenge}J\lozenge_j\Phi_p)\\ &=-\langle R,J\lozenge_k\lozenge_j\Phi_p\rangle-\langle J\lozenge_k\Phi_p,\lozenge_j\Phi_p\rangle=-\langle R,J\lozenge_k\lozenge_j\Phi_p\rangle,\end{aligned}$$

We have

$$\begin{aligned}\partial_{p_k}\mathcal{G}_j&=\Omega(e^{J\tau\cdot\lozenge}R,e^{J\tau\cdot\lozenge}J\lozenge_j\partial_{p_k}\Phi_p)-\Omega(e^{J\tau\cdot\lozenge}\partial_{p_k}\Phi_p e^{J\tau\cdot\lozenge}J\lozenge_j\Phi_p)\\ &=-\langle R,\lozenge_j\partial_{p_k}\Phi_p\rangle+\langle\partial_{p_k}\Phi_p,\lozenge_j\Phi_p\rangle=-\langle R,\lozenge_j\partial_{p_k}\Phi_p\rangle+\delta_{jk}.\end{aligned}$$

At $U=\Psi_0$, $\tau=\tau_0$ and $p=p_0$ we have $\mathcal{F}_j=\mathcal{G}_j=0$. Since in this case $R=0$ we get the desired result by the Implicit Function Theorem. □

2.2 Spectral coordinates

Lemmas 2.4–2.2 lead to a natural decomposition of (2.3). To write it we need further notation.

We are ready for the natural coordinates decomposition. Let $\Pi(U_0) = p_0$. We consider for $R \in N_g^{\perp}(\mathcal{H}_{p_0}^*)$ the map

$$(\tau, p, R) \to U = e^{J\tau\cdot\Diamond}(\Phi_p + P(p)R). \tag{2.23}$$

We have the following formulas,

$$\frac{\partial}{\partial \tau_j} = J\Diamond_j U \ , \quad \frac{\partial}{\partial p_j} = e^{J\tau\cdot\Diamond}(\partial_{p_j}\Phi_p + \partial_{p_j}P(p)R), \tag{2.24}$$

with $\frac{\partial}{\partial p_j} \in C^{\infty}(\mathcal{U} \cap \Sigma_k, \Sigma_{k'})$ for any pair $(k, k') \in \mathbb{N}^2$, with $\mathcal{U} \subset \Sigma_{-n}$ the neighborhood of $e^{J\tau_0\cdot\Diamond}\Phi_{p_0}$ in Lemma 2.4. Similarly, $\frac{\partial}{\partial \tau_j} \in C^0(\mathcal{U} \cap \Sigma_k, \Sigma_{k-d_j})$. We have what follows.

Lemma 2.5. *Consider the $n \geq 0$ and $\mathcal{U}$ in Lemma 2.4 and fix an integer $k \geq -n$. Then the map $U \to R(U) = R$ is $C^0(\mathcal{U} \cap \Sigma_k, \Sigma_k)$. For $k \geq -n + \boldsymbol{d}$ we have $R \in C^1(\mathcal{U} \cap \Sigma_k, \Sigma_{k-\boldsymbol{d}})$. For $\mathcal{U}$ sufficiently small in Σ_{-n} the Frechét derivative $R'(U)$ of $R(U)$ is defined by the following formula, summing on the repeated index j,*

$$R'(U) = (P(p)P(p_0))^{-1}P(p)\big[e^{-J\tau\cdot\Diamond} \ \mathbb{1} - J\Diamond_j P(p)R \, d\tau_j - \partial_{p_j}P(p)R \, dp_j\big],$$

where $(P(p)P(p_0))^{-1} : N_g^{\perp}(\mathcal{H}_p^*) \cap \Sigma_{k-\boldsymbol{d}} \to N_g^{\perp}(\mathcal{H}_{p_0}^*) \cap \Sigma_{k-\boldsymbol{d}}$ is well defined by Lemma 2.3.

Proof. The continuity of $R(U)$ follows from $R = e^{-J\tau\cdot\Diamond}U - \Phi_p$ and

$$\begin{aligned}&R - R' = e^{-J\tau\cdot\Diamond}U - e^{-J\tau'\cdot\Diamond}U' + \Phi_{p'} - \Phi_p = \\ &\Phi_{p'} - \Phi_p + (e^{-J\tau\cdot\Diamond} - e^{-J\tau'\cdot\Diamond})U + e^{-J\tau'\cdot\Diamond}(U - U').\end{aligned}$$

Then use $p \to \Phi_p \in C^{\infty}(\mathcal{O}, \mathcal{S})$, the fact that $e^{J\tau\cdot\Diamond}$ is strongly continuous in Σ_k and locally uniformly bounded therein. The fact that $R(U)$ has Frechét derivative follows by the chain rule. To get the formula for $R'(U)$ notice that the equalities $R'\frac{\partial}{\partial p_j} = R'\frac{\partial}{\partial \tau_j} = 0$ and $R'e^{J\tau\cdot\Diamond}P(p)P(p_0) = \mathbb{1}_{|N_g^{\perp}(\mathcal{H}_{p_0}^*)}$ characterize R'. We claim we have

$$R' = \mathbf{a}_j d\tau_j + \mathbf{b}_j dp_j + (P(p)P(p_0))^{-1}P(p)e^{-J\tau\cdot\Diamond} \tag{2.25}$$

for some $\mathbf{a}_j$ and $\mathbf{b}_j$. First of all, by the independence of coordinates (τ, p) from $R \in N_g^{\perp}(\mathcal{H}_{p_0}^*)$,

$$d\tau_j \circ e^{J\tau\cdot\Diamond}P(p)P(p_0) = dp_j \circ e^{J\tau\cdot\Diamond}P(p)P(p_0) = 0.$$

Indeed for $g \in N_g^{\perp}(\mathcal{H}^*_{p_0})$ we have for instance

$$0 = \frac{d}{dt}\tau_j(u(\tau,p,R+tg))_{|t=0} = \frac{d}{dt}\tau_j(e^{J\tau\cdot\Diamond}(\Phi_p + P(p)P(p_0)(R+tg)))_{|t=0}$$
$$= d\tau_j \circ e^{J\tau\cdot\Diamond}P(p)P(p_0)g.$$

Secondarily, by the definition of $(P(p)P(p_0))^{-1}$,

$$(P(p)P(p_0))^{-1}P(p)e^{-J\tau\cdot\Diamond} \circ e^{J\tau\cdot\Diamond}P(p)P(p_0) = 1\!\!1_{N_g^{\perp}(\mathcal{H}^*_{p_0})}.$$

Hence we get the claimed equality (2.25).
To get $\mathbf{a}_j$ and $\mathbf{b}_j$ notice that by $R'\frac{\partial}{\partial\tau_j} = 0$ and $P(p)J\Diamond_j\Phi_p = 0$

$$\begin{aligned}&\mathbf{a}_j = -(P(p)P(p_0))^{-1}P(p)e^{-J\tau\cdot\Diamond}\frac{\partial}{\partial\tau_j} = \\ &-(P(p)P(p_0))^{-1}P(p)e^{-J\tau\cdot\Diamond}e^{J\tau\cdot\Diamond}J\Diamond_j(\Phi_p + P(p)R) = \\ &-(P(p)P(p_0))^{-1}P(p)J\Diamond_jP(p)R.\end{aligned}$$

Similarly by $R'\frac{\partial}{\partial p_j} = 0$ and $P(p)\partial_{p_j}\Phi_p = 0$

$$\begin{aligned}&\mathbf{b}_j = -(P(p)P(p_0))^{-1}P(p)e^{-J\tau\cdot\Diamond}\frac{\partial}{\partial p_j} = \\ &-(P(p)P(p_0))^{-1}P(p)(\partial_{p_j}\Phi_p + \partial_{p_j}P(p)R) = \\ &-(P(p)P(p_0))^{-1}P(p)\partial_{p_j}P(p)R.\end{aligned}$$

□

A crucial point in the stability proofs in [3, 4, 6, 7], first realized and used in [8], is the importance not to loose track of the Hamiltonian nature of (2.3), in whichever coordinates the system is written. Thus we have what follows.

Lemma 2.6. *In the coordinate system* (2.23)*, system* (2.3) *can be written as*

$$\dot{p} = \{p, E\}\,,\ \dot{\tau} = \{\tau, E\}\,,\ \dot{R} = \{R, E\}. \tag{2.26}$$

Proof. The statement is not standard only for $\dot{R} = \{R, E\}$. Notice that (2.3) can be written as

$$\begin{aligned}\dot{U} &= J\dot{\tau}\cdot\Diamond U + e^{J\tau\cdot\Diamond}\dot{p}\cdot\nabla_p(\Phi_p + P(p)R) + e^{J\tau\cdot\Diamond}P(p)\dot{R} \\ &= \sum_j \dot{\tau}_j\frac{\partial}{\partial\tau_j} + \dot{p}_j\frac{\partial}{\partial p_j} + e^{J\tau\cdot\Diamond}P(p)\dot{R} = J\nabla E(U).\end{aligned} \tag{2.27}$$

When we apply the derivative $R'(U)$ to (2.27), all the terms in the lhs of the last line cancel except for

$$R'(U)e^{J\tau\cdot\Diamond}P(p)\dot{R} = R'(U)J\nabla E(U) = R'(U)X_E(U) = \{R, E\},$$

from the definition of hamiltonian field and of Poisson bracket. Finally we use

$$R'(U)e^{J\tau\cdot\Diamond}P(p)\dot{R} = \frac{d}{ds}_{|s=0}R(U(\tau,p,R+s\dot{R})) = \frac{d}{ds}_{|s=0}(R+s\dot{R}) = \dot{R}.$$

□

2.3 Reduction of order of system (2.26)

The following Poisson bracket identities are useful.

Lemma 2.7. *Consider the functions Π_j. Then $X_{\Pi_j} = \frac{\partial}{\partial \tau_j}$. In particular*

$$\{\Pi_j, \tau_k\} = -\delta_{jk} \ , \quad \{\Pi_j, p_k\} \equiv 0 \ , \quad \{R, \Pi_j\} = 0. \tag{2.28}$$

Proof. (2.28) follows from the first claim, which is a consequence of (2.24):

$$X_{\Pi_j}(U) = J\nabla\Pi_j(U) = J\Diamond_j U = \frac{\partial}{\partial \tau_j}.$$

$\square$

We introduce now a new Hamiltonian:

$$K(U) := E(U) - E\left(\Phi_{p_0}\right) - \lambda_j(p)\left(\Pi_j(U) - \Pi_j(U_0)\right). \tag{2.29}$$

Notice that $K(e^{J\tau\cdot\Diamond}U) \equiv K(U)$. Equivalently, $\partial_{\tau_j} K \equiv 0$. We know that for solutions of (2.3) we have $\Pi_j(U(t)) = \Pi_j(U_0)$ and

$$\{p_j, K\} = \{p_j, E\}, \{R, K\} = \{R, E\}, \{\tau_j, K\} = \{\tau_j, E\} + \lambda_j(p).$$

By $\partial_{\tau_j} K \equiv 0$, the evolution of the variables p, R is unchanged if we consider the following new Hamiltonian system:

$$\dot{p}_j = \{p_j, K\}, \quad \dot{\tau}_j = \{\tau_j, K\}, \quad \dot{R} = \{R, K\}. \tag{2.30}$$

It is elementary that the *momenta* $\Pi_j(U)$ are invariants of motion of (2.30).

Before exploiting the invariance of $\Pi_j(U)$ to reduce the order of the system, we introduce appropriate notation. First of all we set

$$\begin{aligned} \mathcal{P}^r &:= \mathbb{R}^{n_0} \times (\Sigma_r \cap N_g^{\perp}(\mathcal{H}_{p_0})) = \{(\tau, R)\}, \\ \widetilde{\mathcal{P}}^r &:= \mathbb{R}^{n_0} \times \mathcal{P}^r = \{(\Pi, \tau, R)\}. \end{aligned} \tag{2.31}$$

We set $\mathcal{P} = \mathcal{P}^0$ and $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}^0$.

Definition 2.8. We will say that $F(t, \varrho, R) \in C^M(I \times \mathcal{A}, \mathbb{R})$ with I a neighborhood of 0 in $\mathbb{R}$ and $\mathcal{A}$ a neighborhood of 0 in $\mathcal{P}^{-K}$ is $\mathcal{R}^{i,j}_{K,M}$ and we will write $F = \mathcal{R}^{i,j}_{K,M}$, or more specifically $F = \mathcal{R}^{i,j}_{K,M}(t, \varrho, R)$, if there exists a $C > 0$ and a smaller neighborhood $\mathcal{A}'$ of 0 s.t.

$$|F(t, \varrho, R)| \le C\|R\|^j_{\Sigma_{-K}} (\|R\|_{\Sigma_{-K}} + |\varrho|)^i \text{ in } I \times \mathcal{A}'. \tag{2.32}$$

We say $F = \mathcal{R}^{i,j}_{K,\infty}$ if $F = \mathcal{R}^{i,j}_{K,m}$ for all $m \ge M$. We say $F = \mathcal{R}^{i,j}_{\infty,M}$ if for all $k \ge K$ the above F is the restriction of an $F(t, \varrho, R) \in C^M(I \times \mathcal{A}_k, \mathbb{R})$ with $\mathcal{A}_k$ a neighborhood of 0 in $\mathcal{P}^{-k}$ and which is $F = \mathcal{R}^{i,j}_{k,M}$. Finally we say $F = \mathcal{R}^{i,j}$ if $F = \mathcal{R}^{i,j}_{K,\infty}$ and $F = \mathcal{R}^{i,j}_{\infty,M}$.

Definition 2.9. We will say that an $T(t,\varrho,R) \in C^M(I\times\mathcal{A},\Sigma_K(\mathbb{R}^3,\mathbb{R}^{2N}))$, with $I\times\mathcal{A}$ like above, is $\mathbf{S}^{i,j}_{K,M}$ and we will write $T=\mathbf{S}^{i,j}_{K,M}$ or more specifically $T=\mathbf{S}^{i,j}_{K,M}(t,\varrho,R)$, if there exists a $C>0$ and a smaller neighborhood $\mathcal{A}'$ of 0 s.t.

$$\|T(t,\varrho,R)\|_{\Sigma_K} \le C\|R\|^j_{\Sigma_{-K}}(\|R\|_{\Sigma_{-K}}+|\varrho|)^i \text{ in } I\times\mathcal{A}'. \tag{2.33}$$

We use notation $T=\mathbf{S}^{i,j}$, $T=\mathbf{S}^{i,j}_{K,\infty}$ or $T=\mathbf{S}^{i,j}_{\infty,M}$ as above.

These notions will be often used also for functions $F=\mathcal{R}^{i,j}_{K,M}(\varrho,R)$ and $T=\mathbf{S}^{i,j}_{K,M}(\varrho,R)$ independent of t.

Remark 2.10. We will see later that the coefficients of the vector fields whose flows are used to change coordinates are symbols as of Definitions 2.8 and 2.9. The definitions of the symbols $\mathcal{R}^{i,j}$ and $\mathbf{S}^{i,j}$ in Def. 3.9 and 3.10 [2] are very restrictive, since they require for the symbols to be defined in the whole $I\times\mathcal{S}'$. The proofs in [2] at most prove that the coefficients of the vector fields in fact are symbols of the form $\mathcal{R}^{i,j}_{K,M}$ and $\mathbf{S}^{i,j}_{K,M}$ in our sense. As an example we refer to Lemmas 3.26 and 5.5 in [2]. In Lemma 3.26 [2] the fact that the b_i and the $\langle W^l;Y\rangle$ are symbols of the form $\mathcal{R}^{j,k}$ for some (j,k) in the sense of Def. 3.10 in [2], requires preliminarily to show at least that they are functions of (ϱ,R) for (ϱ,R) in some neighborhood $\mathcal{U}$ of $(0,0)$ in $\mathbb{R}^{n_0}\times\mathcal{S}'$. This is not addressed in [2] and is far from trivial, since the coefficients of the linear system right above formula (3.60) are are unbounded in any such $\mathcal{U}$. The justification that the coefficients $\Phi_{\mu\nu}(M)$ of χ in Sect. 5 in [2] are in $\mathcal{S}$ is similarly inconclusive. The key step should be that the homological equation in Lemma 5.5 can be solved for all parameters k uniformly in the variable $M\in\mathbb{R}^n$, at least for $|M|<a$ for a fixed a. But the homological equations involve the perturbation of an operator and in [2] the perturbation is not fully analyzed. For example there is no discussion of the norm $\|V_M-V_0\|_{\mathcal{W}^k\to\mathcal{W}^k}$ as k grows and $|M|<a$. This norm should be expected to grow and become large, possibly breaking down the proof of $\Phi_{\mu\nu}(M)\in\mathcal{S}$. In fact it is plausible that $\Phi_{\mu\nu}(M)\in\mathcal{S}$ only for $M=0$.

From the above remarks we can see that no coordinate change in the Birkhoff or in the Darboux steps in [2] is shown to be an *almost smooth* transformation in the sense of Definition 3.15 in [2]. So for instance the proof of the Birkhoff normal forms, that is Theor. 5.2 [2], is inconclusive. The proof of the Darboux step, that is Theor. 3.21 [2], is even sketchier and is similarly inconclusive.

We proceed now to a reduction of order in (2.30). Write

$$\begin{aligned}&\Pi_j(U)=\Pi_j(e^{J\tau\cdot\Diamond}(\Phi_p+P(p)R))=\Pi_j(\Phi_p+P(p)R)\\&=\frac{1}{2}\langle\Diamond_j(\Phi_p+P(p)R),\Phi_p+P(p)R=p_j+\Pi_j(P(p)R)\\&=p_j+\Pi_j(R)+\Pi_j((P(p)-P(p_0))R)+\langle R,\Diamond_j(P(p)-P(p_0))R\rangle.\end{aligned}\tag{2.34}$$

We well move from variables (τ,p,R) to variables (τ,Π,R). Setting $\varrho_j=\Pi_j(R)$, we have

$$p_j = \Pi_j - \varrho_j + \widetilde{\Psi}_j(p - p_0, R) \tag{2.35}$$

with $\widetilde{\Psi}_j = \mathcal{R}^{0,2}(p - p_0, R)$. The implicit function theorem yields:

Lemma 2.11. *There are functions $p_j = p_j(\Pi, \Pi(R), R)$ defined implicitly by (2.34),or (2.35), such that $p_j = \Pi_j - \varrho_j + \Psi_j(\Pi, \varrho, R)$ with $\Psi(p_0, \varrho, R) = \mathcal{R}^{0,2}(\varrho, R)$.*

We consider now (τ, Π, R) as a new coordinate system. By $\frac{\partial}{\partial \tau_k}\Pi_j(U) \equiv 0$ it follows that the vectorfields $\frac{\partial}{\partial \tau_k}$ are the same for the two systems of coordinates. In the new variables, system (2.30) reduces to the pair of systems

$$\dot{\tau}_j = \{\tau_j, K\}\,, \quad \dot{\Pi}_j = 0\,, \tag{2.36}$$

$$\dot{R} = \{R, K\}. \tag{2.37}$$

System (2.37) is closed because of $\partial_{\tau_j} K = 0$.

3 Darboux Theorem

In this section we present one of the two main results of this paper. We seek to reproduce Moser's proof of the Darboux theorem. Specifically we look for a vector field $\mathcal{X}^t$ that will produce a flow as in (3.42) below. The proof of the existence and properties of $\mathcal{X}^t$ is similar to [7], but influenced by the choice of coordinates in [2]. We also add material to justify, once $\mathcal{X}^t$ has been found, the formal formula (3.42). Notice that for [4, 7] formula (3.42) does not require justification because $\mathcal{X}^t$ is a smooth vectorfield on a given manifold. But the situation in [6, 2] is different since now $\mathcal{X}^t$ is not a standard vectorfield on a manifold and Ω is not a regular differential form on the same manifold, so Lie derivative, pullbacks, push forwards and the related differentiation formulas, require justification.

Notice that, to be useful in the asymptotic stability theory, the change of variables has to be such that the new Hamiltonian equations is semilinear. This is why even in [4, 7], where we could apply the standard Darboux theorem for strong symplectic forms on Banach manifolds, see [1] Ch. 9, it is important to select $\mathcal{X}^t$ with an *ad hoc* process.

3.1 Search of a vectorfield

Recall that $\Omega = \langle J^{-1}\ ,\ \rangle$ and consider

$$\Omega_0 := d\tau_j \wedge d\Pi_j + \langle J^{-1}R', R'\rangle. \tag{3.1}$$

Lemma 3.1. *At the points $e^{J\tau\cdot\Diamond}\Phi_{p_0}$ for all $\tau \in \mathbb{R}^{n_0}$ we have $\Omega_0 = \Omega$.*

Consider the following forms:

$$\mathrm{B}_0 := \tau_j d\Pi_j + \frac{1}{2}\langle J^{-1}R, R'\rangle; \quad \mathrm{B} := \mathrm{B}_0 + \alpha \textit{ for} \tag{3.2}$$

$$\begin{aligned}
&\alpha := -\beta_j(p,R)d\Pi_j + \langle \Gamma(p)R + \beta_j(p,R)P^*(p)\Diamond_j P(p)R, R'\rangle \ , \\
&\Gamma(p) := \frac{1}{2}J^{-1}\left(P(p) - P(p_0)\right) \ , \\
&\beta_j(p,R) := \frac{1}{2}\ \frac{\langle P^*(p)J^{-1}R, \partial_{p_j}P(p)R\rangle}{1 + \langle \Diamond_j P(p)R, \partial_{p_j}P(p)R\rangle} \ .
\end{aligned} \tag{3.3}$$

Then $d\mathrm{B}_0 = \Omega_0$ *and* $d\mathrm{B} = \Omega$.

Proof. $d\mathrm{B}_0 = \Omega_0$ follows from the definition of exterior differential. Set $\widetilde{B} := \frac{1}{2}\langle J^{-1}U, \ \rangle$. Notice that $d\widetilde{B} = \Omega$. By (2.23) we get:

$$\widetilde{B}(X) = \frac{1}{2}\langle J^{-1}e^{J\tau\cdot\Diamond}\Phi_p, X\rangle + \frac{1}{2}\langle J^{-1}P(p)R, e^{-J\tau\cdot\Diamond}X\rangle. \tag{3.4}$$

Set $\psi(U) := \frac{1}{2}\langle J^{-1}e^{J\tau\cdot\Diamond}\Phi_p, U\rangle$. Then we claim

$$d\psi = \frac{1}{2}\langle J^{-1}e^{J\tau\cdot\Diamond}\Phi_p, \ \rangle + p_j d\tau_j,$$

where in this proof we will sum on repeated indexes. The last formula implies

$$\widetilde{B} = d\psi - p_j d\tau_j + \frac{1}{2}\langle J^{-1}P(p)R, e^{-J\tau\cdot\Diamond} \ \rangle. \tag{3.5}$$

The desired formula on $d\psi$ follows by

$$\begin{aligned}
&d\psi = \frac{1}{2}\langle J^{-1}e^{J\tau\cdot\Diamond}\Phi_p, \ \rangle + \frac{1}{2}\langle e^{J\tau\cdot\Diamond}\Diamond_j\Phi_p, U\rangle d\tau_j \\
&+ \frac{1}{2}\langle e^{J\tau\cdot\Diamond}J^{-1}\partial_{p_j}\Phi_p, U\rangle dp_j = \frac{1}{2}\langle J^{-1}e^{J\tau\cdot\Diamond}\Phi_p, \ \rangle + \\
&\frac{1}{2}\langle\Diamond_j\Phi_p, \Phi_p + P(p)R\rangle d\tau_j + \frac{1}{2}\langle J^{-1}\partial_{p_j}\Phi_p, \Phi_p + P(p)R\rangle dp_j \overset{\text{by (2.17)}}{=} \\
&\frac{1}{2}\langle J^{-1}e^{J\tau\cdot\Diamond}\Phi_p, \ \rangle + \underbrace{\frac{1}{2}\langle\Diamond_j\Phi_p, \Phi_p\rangle}_{p_j} d\tau_j + \frac{1}{2}\underbrace{\langle J^{-1}\partial_{p_j}\Phi_p, \Phi_p\rangle}_{0 \text{ by (2.14)}} dp_j.
\end{aligned}$$

By Lemma 2.5 and using $P(p)^*J^{-1} = J^{-1}P(p)$ we have

$$\begin{aligned}
&\frac{1}{2}\langle J^{-1}P(p)R, e^{-J\tau\cdot\Diamond} \ \rangle = \frac{1}{2}\langle J^{-1}R, P(p)R' \ \rangle \\
&+ \frac{1}{2}\langle J^{-1}R, P(p)J\Diamond_j P(p)R\rangle d\tau_j + \frac{1}{2}\langle J^{-1}R, P(p)\partial_{p_j}P(p)R\rangle dp_j \\
&= \frac{1}{2}\langle J^{-1}R, R' \ \rangle + \frac{1}{2}\langle J^{-1}R, (P(p) - P(p_0))R' \ \rangle \\
&- \Pi_j(P(p)R)d\tau_j + \frac{1}{2}\langle J^{-1}R, P(p)\partial_{p_j}P(p)R\rangle dp_j.
\end{aligned}$$

So by (3.5) and using $P(p)J = JP^*(p)$ we get

$$\begin{aligned}
&\widetilde{B} - d\psi = -(\overbrace{p_j + \Pi_j(P(p)R)}^{\Pi_j})d\tau_j + \frac{1}{2}\langle J^{-1}R, R' \ \rangle \\
&+ \frac{1}{2}\langle J^{-1}R, (P(p) - P(p_0))R' \ \rangle - \frac{1}{2}\langle P^*(p)J^{-1}R, \partial_{p_j} P(p)R\rangle dp_j.
\end{aligned}$$

Then $d\alpha = \Omega - \Omega_0$ for

$$\begin{aligned}
&\alpha := \widetilde{B} - d\psi - B_0 + d(\Pi_j\tau_j) = \\
&\frac{1}{2}\langle J^{-1}R, (P(p) - P(p_0))R' \ \rangle - \frac{1}{2}\langle P^*(p)J^{-1}R, \partial_{p_j} P(p)R\rangle dp_j.
\end{aligned}$$

By $p_j = \Pi_j - \Pi_j(P(p)R)$ we get

$$dp_j = d\Pi_j - \langle \Diamond_j P(p)R, P(p)R'\rangle - \langle \Diamond_j P(p)R, \partial_{p_j} P(p)R\rangle dp_j.$$

Then inserting the next formula in the formula for α, we obtain (3.3):

$$dp_j = \frac{d\Pi_j - \langle \Diamond_j P(p)R, P(p)R'\rangle}{1 + \langle \Diamond_j P(p)R, \partial_{p_j} P(p)R\rangle}. \tag{3.6}$$

$\square$

In the Lemmas 3.2–3.6 we will initially consider the regularity of the functions in terms of the coordinates (τ, p, R).

Lemma 3.2. *We have $\beta_j \in C^\infty(\mathcal{O} \times \Sigma_{-n}, \mathbb{R})$ for any n. For any pair (n, n') we have $\Gamma \in C^\infty(\mathcal{O}, B(\Sigma_{-n'}, \Sigma_n))$. Summing on repeated indexes, we have*

$$\begin{aligned}
&d\alpha = -\partial_{p_k}\beta_j dp_k \wedge d\Pi_j - \langle \nabla_R \beta_j, R'\rangle \wedge d\Pi_j \\
&+ dp_k \wedge \langle \partial_{p_k}[\Gamma(p)R + \beta_j(p, R)P^*(p)\Diamond_j P(p)R], R'\rangle \\
&+ \langle \nabla_R \beta_j, R'\rangle \wedge \langle P^*(p)\Diamond_j P(p)R, R'\rangle + 2\langle \Gamma R', R'\rangle.
\end{aligned} \tag{3.7}$$

Proof. Follows from a simple computation. In particular, for a $\mathbf{L} \in B(\Sigma_1, L^2)$ fixed, we use the formula

$$\begin{aligned}
&d\langle \mathbf{L}R, R'\rangle(X, Y) := X\langle \mathbf{L}R, R'Y\rangle - Y\langle \mathbf{L}R, R'X\rangle - \langle \mathbf{L}R, R'[X, Y]\rangle \\
&= \langle \mathbf{L}R'X, R'Y\rangle - \langle \mathbf{L}R'Y, R'X\rangle.
\end{aligned}$$

$\square$

Lemma 3.3. *Summing on repeated indexes, we have*

$$\begin{aligned}
d\alpha = {}& \widehat{\delta}_k \partial_{p_k}\beta_j d\Pi_j \wedge d\Pi_k + \langle \widehat{\Gamma}_j + (\widehat{\delta}_k \partial_{p_k}\beta_j - \widehat{\delta}_j \partial_{p_j}\beta_k)\Diamond_k P(p)R, R'\rangle \wedge d\Pi_j \\
&+ 2\langle \Gamma(p)R', R'\rangle + \langle \widetilde{\beta}_j, R'\rangle \wedge \langle P^*(p)\Diamond_j P(p)R, R'\rangle \ ,
\end{aligned}$$

where we have (this time not summing on repeated indexes)

$$
\begin{aligned}
\widehat{\delta}_k &:= \frac{1}{1+\langle \Diamond_k P(p)R, \partial_{p_k} P(p)R\rangle}\ ,\\
\widehat{\Gamma}_j &:= -\nabla_R \beta_j - \widehat{\delta}_j[\partial_{p_j}\Gamma R + \sum_{i=1}^{n_0} \beta_i \partial_{p_j}\left(P^*(p)\Diamond_i P(p)\right) R]\\
&\quad + \sum_{k=1}^{n_0} (\widehat{\delta}_k \partial_{p_k}\beta_j - \widehat{\delta}_j \partial_{p_j}\beta_k)(P^*(p)-1)\Diamond_k P(p)R\\
\widetilde{\beta}_j &:= \nabla_R \beta_j + \widehat{\delta}_j \partial_{p_j}(\Gamma + \sum_{k=1}^{n_0} \beta_k P^*(p)\Diamond_k P(p))R.
\end{aligned}
$$

Proof. Follows by an elementary computation substituting (3.6) in (3.7) $\square$

Lemma 3.4. *For any fixed large n and for $\varepsilon_0 > 0$, consider the set $\mathcal{U}_{\mathbf{d}} \subset \widetilde{\mathcal{P}}^{\boldsymbol{d}} = \{(p,R)\}$ defined by $\|R\|_{\Sigma_{-n}} \le \varepsilon_0$ and $|p-p_0| \le \varepsilon_0$. Then for ε_0 small enough there exists a unique vectorfield $\mathcal{X}^t : \mathcal{U}_{\mathbf{d}} \to \widetilde{\mathcal{P}}$ which solves $i_{\mathcal{X}^t}\Omega_t = -\alpha$, where $\Omega_t := \Omega_0 + t(\Omega - \Omega_0)$.*

Proof. First of all we consider Y such that $i_Y\Omega_0 = -\alpha$, that is to say

$$
\begin{aligned}
&(Y)_{\tau_j} d\Pi_j - (Y)_{\Pi_j} d\tau_j + \langle J^{-1}(Y)_R, R'\rangle\\
&= \beta_j(p,R) d\Pi_j - \langle \Gamma(p)R + \beta_j(p,R)P^*(p)\Diamond_j P(p)R, R'\rangle\,.
\end{aligned}
$$

This yields

$$
\begin{aligned}
&(Y)_{\tau_j} = \beta_j(p,R) = \mathcal{R}^{0,2}(p,R)\ , \quad (Y)_{\Pi_j} = 0\ ,\\
&(Y)_R = -P(p_0)J\Gamma(p)R - \beta_j(p,R)P(p_0)JP^*(p)\Diamond_j P(p)R\\
&= \mathbf{S}^{1,1}(p-p_0,R) + \mathcal{R}^{0,2}(p,R)P(p_0)P(p)J\Diamond_j P(p)R.
\end{aligned}
\tag{3.8}
$$

Equation $i_{\mathcal{X}^t}\Omega_t = -\alpha$ is equivalent to

$$
(1+t\mathcal{K})\mathcal{X}^t = Y \tag{3.9}
$$

where the operator $\mathcal{K}$ is defined by $i_X d\alpha = i_{\mathcal{K}X}\Omega_0$. In coordinates, (3.9) becomes $(\mathcal{X}^t)_{\Pi_j} = 0$ and, for $P = P(p)$,

$$
(\mathcal{X}^t)_{\tau_j} + t\langle \widehat{\Gamma}_j + (\widehat{\delta}_k \partial_{p_k}\beta_j - \widehat{\delta}_j \partial_{p_j}\beta_k)\Diamond_k PR, (\mathcal{X}^t)_R\rangle = -\beta_j, \tag{3.10}
$$

$$
(\mathcal{X}^t)_R + t\mathcal{L}(\mathcal{X}^t)_R = (Y)_R \text{ ,where for } X \in N_g^{\perp}(\mathcal{H}^*_{p_0}) \tag{3.11}
$$

$$
\mathcal{L}X := P(p_0)J\left[2\Gamma X + \langle \widetilde{\beta}_j, X\rangle P^*\Diamond_j PR - \langle P^*\Diamond_j PR, X\rangle \widetilde{\beta}_j\right]. \tag{3.12}
$$

(3.12) implies the following lemma.

Lemma 3.5. *We have, summing on repeated indexes, with i varying in some finite set,*

$$
\mathcal{L}X = \mathcal{A}_j(X)J\Diamond_j R + \mathcal{B}_i(X)\Psi_i \tag{3.13}
$$

where: $\Psi_i = \mathbf{S}^{0,0}(p-p_0,R)$*; for* $L = \mathcal{A}_j, \mathcal{B}_i$*, we have* $L \in C^\infty(\mathcal{U}_{\mathbf{d}}, B(L^2,\mathbb{R}))$ *with*

$$L(X) = L_j \langle \lozenge_j R, X\rangle + \langle \widetilde{L}, X\rangle, \tag{3.14}$$

where we have $\widetilde{L} = \mathbf{S}^{1,0}(p-p_0,R)$ *and* $L_j \in \mathcal{R}^{0,0}(p-p_0,R)$*.*

Proof. Schematically, for $\widetilde{L}_i = \mathbf{S}^{0,0}(p-p_0,R)$ and $\Psi_i = \mathbf{S}^{0,0}(p-p_0,R)$ we have

$$\begin{aligned}
&P(p)R = R - P_{N_g}(p)R = R + \sum_i \langle \widetilde{L}_i, R\rangle \Psi_i\,,\\
&P^*(p)\lozenge_k R = \lozenge_k R - P^*_{N_g}(p)\lozenge_k R = \lozenge_k R + \sum_i \langle \widetilde{L}_i, R\rangle \Psi_i.
\end{aligned}$$

Then $(P^*(p)\lozenge_k P(p) - \lozenge_k)R = \mathbf{S}^{0,1}(p-p_0,R)$.
By the definition of $\widetilde{\beta}_j$ we have

$$\begin{aligned}
&\widetilde{\beta}_j = \sum_k \widehat{\delta}_j(\partial_{p_j}\beta_k)\lozenge_k R + \widehat{L}\\
&\widehat{L} := \nabla_R \beta_j + \frac{1}{2}J^{-1}\widehat{\delta}_j \partial_{p_j} P(p)R + \sum_k \beta_k \partial_{p_j}(P^*(p)\lozenge_k P(p))R\\
&- \sum_k \widehat{\delta}_j \partial_{p_j}\beta_k \left[P^*_{N_g}(p)\lozenge_k P(p)R + \lozenge_k P_{N_g}(p)R \right],
\end{aligned}$$

where $\widehat{L} = \mathbf{S}^{0,1}(p-p_0,R)$.
We also have $\Gamma X = \frac{1}{2}J^{-1}(P_{N_g}(p_0) - P_{N_g}(p))X = \sum_i \langle \widetilde{L}_i, X\rangle \Psi_i$ with $\widetilde{L}_i = \mathbf{S}^{1,0}(p-p_0,R)$ and $\Psi_i = \mathbf{S}^{0,0}(p-p_0,R)$. This yields the result.

□

Lemma 3.6. *System* (3.10)–(3.12) *admits exactly one solution* $\mathcal{X}^t$*. For* $\mathcal{A}_j = \mathcal{R}^{0,2}_{n,\infty}(t,p-p_0,R)$*,* $\mathcal{D} = \mathbf{S}^{1,1}_{n,\infty}(t,p-p_0,R)$ *with* $|t|<3$*, we have*

$$(\mathcal{X}^t)_R = \mathcal{A}_j J \lozenge_j R + \mathcal{D}. \tag{3.15}$$

Proof. Recall Y defined by $i_Y\Omega_0 = -\alpha$. By (3.8) with $\widetilde{\mathcal{A}}_j = \mathcal{R}^{0,2}_{n,\infty}(p-p_0,R)$ and $\widetilde{\mathcal{D}} = \mathbf{S}^{1,1}_{n,\infty}(p-p_0,R)$ we have $(Y)_R = \widetilde{\mathcal{A}}_j J\lozenge_j R + \widetilde{\mathcal{D}}$. By $(\mathcal{X}^t)_R + t\mathcal{L}(\mathcal{X}^t)_R = (Y)_R$ and Lemma 3.5 this implies for $X = (\mathcal{X}^t)_R$

$$\begin{aligned}
&\langle \lozenge_k R, X\rangle + t\mathcal{B}_i(X)\langle \lozenge_k R, \Psi_i\rangle = \langle \lozenge_k R, (Y)_R\rangle\\
&\langle \widetilde{L}, X\rangle + t\mathcal{A}_j(X)\langle \widetilde{L}, J\lozenge_j R\rangle + t\mathcal{B}_i(X)\langle \widetilde{L}, \Psi_i\rangle = \langle \widetilde{L}, (Y)_R\rangle,
\end{aligned}$$

as L runs through all the $L = \mathcal{A}_j, \mathcal{B}_i$. Taking appropriate linear combinations of these equations with the coefficients L_j of $L = \mathcal{A}_j, \mathcal{B}_i$, see Lemma 3.5, for a matrix $\mathbf{R}^{0,1}(p-p_0,R)$ whose coefficients are $\mathcal{R}^{0,1}(p-p_0,R)$, we get

$$(1 + t\mathbf{R}^{0,1}(p-p_0,R)) \begin{pmatrix} \mathcal{A}_j((\mathcal{X}^t)_R) \\ \mathcal{B}_i((\mathcal{X}^t)_R) \end{pmatrix} = \begin{pmatrix} \mathcal{A}_j((Y)_R) \\ \mathcal{B}_i((Y)_R) \end{pmatrix}.$$

Then we get

$$\begin{pmatrix} \mathcal{A}_j((\mathcal{X}^t)_R) \\ \mathcal{B}_i((\mathcal{X}^t)_R) \end{pmatrix} = (1 + t\mathbf{R}^{0,1}(p - p_0, R))^{-1} \begin{pmatrix} \mathcal{A}_j((Y)_R) \\ \mathcal{B}_i((Y)_R) \end{pmatrix}. \tag{3.16}$$

Using the left hand side of (3.16) set

$$\mathcal{L}(\mathcal{X}^t)_R := \mathcal{A}_j((\mathcal{X}^t)_R)J\Diamond_j R + \mathcal{B}_i((\mathcal{X}^t)_R)\Psi_i. \tag{3.17}$$

The rhs of (3.17) satisfies the properties stated for the rhs of (3.15). Finally set $(\mathcal{X}^t)_R := (Y)_R - t\mathcal{L}(\mathcal{X}^t)_R$. This is a solution of (3.11). It is elementary to see from the argument that such solution is unique and that it satisfies the properties of the statement. $\square$

Turning to coordinates (τ, Π, R) and by Lemma 2.11 we conclude what follows.

Lemma 3.7. *Consider the coordinate system (τ, Π, R). For G any of the $\mathcal{A}_j$, $\mathcal{D}$ in Lemma 3.6, we have $G = G(\Pi, \Pi(R), R)$, with $G(\Pi, \varrho, R)$ smooth w.r.t. $(\Pi, \varrho, R) \in \mathcal{U}_{\boldsymbol{d}}$, with $\mathcal{U}_{\boldsymbol{d}}$ formed by the $(\Pi, \varrho, R) \in \mathbb{R}^{2n_0} \times (\Sigma_{\boldsymbol{d}} \cap N_g^{\perp}(\mathcal{H}_{p_0}))$ defined by the inequalities $\|R\|_{\Sigma_{-n}} \le \varepsilon$, $|\varrho| \le \varepsilon$ and $|\Pi - p_0| \le \varepsilon$ for $\varepsilon > 0$ small enough.*

3.2 Flows

The following lemma is repeatedly used in the sequel, see Lemma 3.24 [2].

Lemma 3.8. *Below we pick $r, M, M_0, s, s', k, l \in \mathbb{N} \cup \{0\}$ with $1 \le l \le M$. Consider a system*

$$\begin{aligned} &\dot{\tau}_j = T_j(t, \Pi, \Pi(R), R) \ , \quad \dot{\Pi}_j = 0 \ , \\ &\dot{R} = \mathcal{A}_j(t, \Pi, \Pi(R), R)J\Diamond_j R + \mathcal{D}(t, \Pi, \Pi(R), R), \end{aligned} \tag{3.18}$$

where we assume what follows.

- $P_{N_g(p_0)}(\mathcal{A}_j J\Diamond_j R + \mathcal{D}) \equiv 0$.
- *At $\Pi = p_0$, dropping the dependence on Π and for $\mathcal{U}_{-r}$ a neighborhood of 0 in $\mathcal{P}^{-r}$, we have $\mathcal{A}(t, \varrho, R) \in C^M((-3,3) \times \mathcal{U}_{-r}, \mathbb{R}^{n_0})$ and $\mathcal{D}(t, \varrho, R) \in C^M((-3,3) \times \mathcal{U}_{-r}, \Sigma_r)$*
- *In $(-3,3) \times \mathcal{U}_{-r}$ for a fixed i in $\{0,1\}$, and a fixed C_r, we have:*

$$\begin{aligned} &|\mathcal{A}(t, \varrho, R)| \le C\|R\|_{\Sigma_{-r}}^{M_0+1}, \\ &\|\mathcal{D}(t, \varrho, R)\|_{\Sigma_r} \le C(|\varrho| + \|R\|_{\Sigma_{-r}})^i \|R\|_{\Sigma_{-r}}^{M_0}. \end{aligned} \tag{3.19}$$

Let $k \in \mathbb{Z} \cap [0, r - (l+1)\boldsymbol{d}]$ and set for $s'' \ge \boldsymbol{d}$ (or $s'' \ge \boldsymbol{d}/2$ if $\boldsymbol{d}/2 \in \mathbb{N}$)

$$\mathcal{U}^{s''}_{\varepsilon_1,k} := \{(\tau, \Pi, R) \in \widetilde{\mathcal{P}}^{s''} \ : \ \Pi = p_0 \ , \ \|R\|_{\Sigma_{-k}} + |\Pi(R)| \le \varepsilon_1\}. \tag{3.20}$$

Then for $\varepsilon_1 > 0$ small enough, the initial value problem associated to (3.18) for $\Pi = p_0$ defines a flow $\mathfrak{F}^t = (\mathfrak{F}^t_\tau, \mathfrak{F}^t_R)$ for $t \in [-2,2]$ in $\mathcal{U}^{\boldsymbol{d}}_{\varepsilon_1,k}$. In particular for $\Pi = p_0$, for R in a neighborhood $B_{\Sigma_{-k}}$ of 0 in Σ_{-k} and $\Pi(R)$ in a neighborhood $B_{\mathbb{R}^{n_0}}$ of 0 in $\mathbb{R}^{n_0}$, we have

$$\mathfrak{F}^t_R(\Pi(R), R) = e^{Jq(t,\Pi(R),R)\cdot\Diamond}(R + \boldsymbol{S}(t,\Pi(R),R)), \tag{3.21}$$

$$\begin{aligned} &\text{with } \boldsymbol{S} \in C^l((-2,2)\times B_{\mathbb{R}^{n_0}} \times B_{\Sigma_{-k}}, \Sigma_{r-(l+1)\boldsymbol{d}}) \\ &\quad q \in C^l((-2,2)\times B_{\mathbb{R}^{n_0}} \times B_{\Sigma_{-k}}, \mathbb{R}^{n_0}). \end{aligned} \tag{3.22}$$

For fixed $C > 0$ we have

$$\begin{aligned} &|q(t,\varrho,R)| \le C\|R\|^{M_0+1}_{\Sigma_{(l+1)\boldsymbol{d}-r}}, \\ &\|\boldsymbol{S}(t,\varrho,R)\|_{\Sigma_{r-(l+1)\boldsymbol{d}}} \le C(|\varrho| + \|R\|_{\Sigma_{(l+1)\boldsymbol{d}-r}})^i \|R\|^{M_0}_{\Sigma_{(l+1)\boldsymbol{d}-r}}. \end{aligned} \tag{3.23}$$

Furthermore we have $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2$ with

$$\begin{aligned} &\boldsymbol{S}_1(t,\Pi(R),R) = \int_0^t \mathcal{D}(t',\Pi(R(t')),R(t'))dt' \\ &\|\boldsymbol{S}_2(t,\varrho,R)\|_{\Sigma_s} \le C\|R\|^{2M_0+1}_{\Sigma_{(l+1)\boldsymbol{d}-r}} (|\varrho| + \|R\|_{\Sigma_{(l+1)\boldsymbol{d}-r}})^i. \end{aligned} \tag{3.24}$$

For $r - (l+1)\boldsymbol{d} \ge s' \ge s + l\mathbf{d} \ge l\mathbf{d}$ and $k \in \mathbb{Z} \cap [0, r-(l+1)\boldsymbol{d}]$ and for $\varepsilon_1 > 0$ sufficiently small, we have

$$\mathfrak{F}^t \in C^l((-2,2)\times \mathcal{U}^{s'}_{\varepsilon_1,k}, \widetilde{\mathcal{P}}^s). \tag{3.25}$$

Furthermore, there exists $\varepsilon_2 > 0$ such that

$$\mathfrak{F}^t(\mathcal{U}^{s'}_{\varepsilon_2,k}) \subset \mathcal{U}^{s'}_{\varepsilon_1,k} \text{ for all } |t| \le 2 \ . \tag{3.26}$$

We have

$$\mathfrak{F}^t(e^{J\tau\cdot\Diamond}U) \equiv e^{J\tau\cdot\Diamond}\mathfrak{F}^t(U). \tag{3.27}$$

Proof. It is enough to focus on the equation for R. Set $S = e^{-Jq\cdot\Diamond}R$ for $q \in \mathbb{R}^{n_0}$. Then consider the following system:

$$\begin{aligned} &\dot S = e^{-Jq\cdot\Diamond}\mathcal{D}(t,\varrho,e^{Jq\cdot\Diamond}S) \ , \\ &\dot q = \mathcal{A}(t,\varrho,e^{Jq\cdot\Diamond}S) \quad , \quad q(0) = 0, \\ &\dot\varrho_j = \langle S, e^{-Jq\cdot\Diamond}\Diamond_j \mathcal{D}(t,\varrho,e^{Jq\cdot\Diamond}S)\rangle \ . \end{aligned} \tag{3.28}$$

For $l \le M$ and $k, s'' \in [0, r-(l+1)\mathbf{d}]$ the field in (3.28) is $C^l((-3,3)\times\mathcal{U}_{-k}, \Sigma_{s''}\times\mathbb{R}^{2n_0})$ with $\mathcal{U}_{-k} \subset \Sigma_{-k}\times\mathbb{R}^{2n_0}$ a neighborhood of the equilibrium 0. This follows from the fact that $(q,X) \to e^{Jq\cdot\Diamond}X$ is in $C^l(\mathbb{R}^{n_0}\times\Sigma_\ell, \Sigma_{\ell-l\mathbf{d}})$ for all $\ell \in \mathbb{Z}$ and from the hypotheses on $\mathcal{A}$ and $\mathcal{D}$. For example

$$(t,q,\varrho,S) \to e^{-Jq\cdot\Diamond}\Diamond_j \mathcal{D}(t,\varrho,e^{Jq\cdot\Diamond}S) \in C^l((-3,3)\times\mathbb{R}^{2n_0}\times\Sigma_{l\mathbf{d}-r}, \Sigma_{r-(l+1)\mathbf{d}}),$$

(more precisely for (q, ϱ, S) in a neighborhood of the origin). So

$$(t, q, \varrho, S) \to \langle S, e^{-Jq\cdot\Diamond}\Diamond_j \mathcal{D}(t, \varrho, e^{Jq\cdot\Diamond}S)\rangle,$$

is in $C^l((-3,3)\times\mathbb{R}^{2n_0}\times\Sigma_{-k},\mathbb{R})$ for $k \le r-(l+1)\mathbf{d}$ (for (q,ϱ,S) near origin). For $l \ge 1$ we can apply to (3.28) standard theory of ODE's to conclude that there are neighborhoods of the origin $B_{\mathbb{R}^{2n_0}} \subset \mathbb{R}^{2n_0}$ and $B_{\Sigma_{-k}} \subset \Sigma_{-k}$ such that the flow is of the form

$$\begin{aligned} &S(t) = R + \mathbf{S}(t,\varrho,R)\ , \quad \mathbf{S}(0,\varrho,R) = 0\ ,\\ &q(t) = q(t,\varrho,R)\ , \quad q(0,\varrho,R) = 0\ ,\\ &\varrho(t) = \varrho + \overline{\varrho}(t,\varrho,R)\ , \quad \overline{\varrho}(0,\varrho,R) = 0\ , \end{aligned} \tag{3.29}$$

$$\begin{aligned} \text{with } &\mathbf{S} \in C^l((-2,2)\times B_{\mathbb{R}^{n_0}}\times B_{\Sigma_{-k}}, \Sigma_{r-(l+1)\mathbf{d}})\\ &\overline{\varrho}, q(t,\varrho,R) \in C^l((-2,2)\times B_{\mathbb{R}^{n_0}}\times B_{\Sigma_{-k}}, \mathbb{R}^{n_0}). \end{aligned} \tag{3.30}$$

For $S \in \Sigma_{\mathbf{d}} \cap B_{\Sigma_{-k}}$ and $S(0) = S$, choosing $s'' \ge \mathbf{d}$ we have $S(t) \in \Sigma_{\mathbf{d}}$ with $\Pi(S(t)) = \varrho(t)$ for $\varrho(0) = \varrho = \Pi(S)$. Then (3.30) yields (3.22) (we can replace $\Sigma_{\mathbf{d}}$ with $\Sigma_{\frac{\mathbf{d}}{2}}$ if $\frac{\mathbf{d}}{2} \in \mathbb{N}$). (3.21) and (3.22) yield (3.25).
We have for $R(0) = R$

$$R(t) = e^{Jq(t)\cdot\Diamond}(R + \int_0^t e^{-Jq(t')\cdot\Diamond}\mathcal{D}(t', \varrho(t'), R(t'))dt'). \tag{3.31}$$

By (A6), for $\epsilon = 0$, and by (3.19), for $|s''| \le r-(l+1)\mathbf{d}$ we have

$$\begin{aligned} &\|R(t)\|_{\Sigma_{s''}} \le C\|R\|_{\Sigma_{s''}} + C\int_0^t \|\mathcal{D}(t',\varrho(t),R(t'))\|_{\Sigma_r}dt'\\ &\le C\|R\|_{\Sigma_{s''}} + C\int_0^t \|R(t')\|^{M_0}_{\Sigma_{-r}}(|\varrho(t')| + \|R(t')\|_{\Sigma_{-r}})^i dt'\\ &\le C\|R\|_{\Sigma_{s''}} + C\int_0^t \|R(t')\|^{M_0}_{\Sigma_{s''}}(|\varrho(t')| + \|R(t')\|_{\Sigma_{s''}})^i dt', \end{aligned} \tag{3.32}$$

with the caveat that the second line is purely formal and is used to get the third line, where the integrand is continuous. Proceeding similarly, for $\varrho(0) = \varrho$

$$\begin{aligned} &|\varrho(t) - \varrho| \le \int_0^t |\langle R(t'), \Diamond\mathcal{D}(t', R(t'), \varrho(t'))\rangle| dt'\\ &\le \int_0^t \|R(t')\|_{\Sigma_{(l+1)\mathbf{d}-r}}\|\mathcal{D}(t',\varrho(t),R(t'))\|_{\Sigma_{r-l\mathbf{d}}}dt'\\ &\le C\int_0^t \|R(t')\|^{M_0+1}_{\Sigma_{(l+1)\mathbf{d}-r}}(|\varrho(t')| + \|R(t')\|_{\Sigma_{(l+1)\mathbf{d}-r}})^i dt'. \end{aligned} \tag{3.33}$$

So for $|s''| \le r-(l+1)\mathbf{d}$, using the continuity in t' of the integrals in the last lines of (3.32) and (3.33), by the Gronwall inequality there is a fixed C such

that for all $|t| \leq 2$ we have

$$\|R(t)\|_{\Sigma_{s''}} \leq C\|R\|_{\Sigma_{s''}}, \tag{3.34}$$
$$|\varrho(t) - \varrho| \leq C\|R\|^{M_0+1}_{\Sigma_{(l+1)\mathbf{d}-r}}(|\varrho| + \|R\|_{\Sigma_{(l+1)\mathbf{d}-r}})^i. \tag{3.35}$$

By (3.34) for $s'' = s'$ and $s'' = -k$ and by $|\varrho(t)-\varrho| \leq C\|R\|^{M_0+1}_{\Sigma_{-k}}(|\varrho|+\|R\|_{\Sigma_{-k}})^i$, we get $\mathfrak{F}^t(\mathcal{U}^{s'}_{\varepsilon_2,k}) \subset \mathcal{U}^{s'}_{\varepsilon_1,k}$ for all $|t| \leq 2$ for $\varepsilon_1 \gg \varepsilon_2$, that is (3.26).
We have

$$S(t, \varrho, R) = \int_0^t e^{-Jq(t')\cdot\Diamond}\mathcal{D}(t', \varrho(t'), R(t'))dt'),$$

Proceeding as for (3.32) and using (3.34)–(3.35) we get the estimate for $\mathbf{S}$ in (3.23). The estimate on q is obtained similarly integrating the second equation in (3.29).
We have

$$\mathbf{S}_2(t, R, \varrho) = \int_0^1 dt'' \int_0^t e^{-t''q(t')\cdot\Diamond}q(t') \cdot \Diamond\mathcal{D}(t', \varrho(t), R(t'))dt' \tag{3.36}$$

Then by (3.34)–(3.35) we get

$$\begin{aligned}
&\|\mathbf{S}_2(t, R, \varrho)\|_{\Sigma_{r-\mathbf{d}}} \leq C'' \int_0^t |q(t')|\|\mathcal{D}(t', \varrho(t), R(t'))\|_{\Sigma_{r-\mathbf{d}}}dt' \\
&\leq C' \int_0^t \|R(t')\|^{2M_0+1}_{\Sigma_{(l+1)\mathbf{d}-r}}(|\varrho(t')| + \|R(t')\|_{\Sigma_{(l+1)\mathbf{d}-r}})^i dt' \\
&\leq C\|R\|^{2M_0+1}_{\Sigma_{(l+1)\mathbf{d}-r}}(|\varrho| + \|R\|_{\Sigma_{(l+1)\mathbf{d}-r}})^i.
\end{aligned} \tag{3.37}$$

This yields (3.24). (3.25) follows by (3.21)–(3.22). Finally, (3.27) follows immediately from (3.21).

□

Lemma 3.9. *Assume hypotheses and conclusions of Lemma 3.8. Consider the flow of system* (3.28) *for* $\Pi = p_0$ *. Denote the flow in the space with variables* $\{(\varrho, R)\}$ *by* $\mathfrak{F}^t = (\mathfrak{F}^t_\varrho, \mathfrak{F}^t_R)$*. Then we have*

$$\begin{aligned}
&\mathfrak{F}^t_R(\varrho, R) = e^{Jq(t,\varrho,R)\cdot\Diamond}(R + \boldsymbol{S}(t, \varrho, R)) \\
&\mathfrak{F}^t_\varrho(\varrho, R) = \varrho + \overline{\varrho}(t, \varrho, R).
\end{aligned} \tag{3.38}$$

Furthermore, the following facts hold.

(1) Let $k \in \mathbb{Z}\cap[0, r-(l+1)\boldsymbol{d}]$ *and* $h \geq \max\{k+l\boldsymbol{d}, (2l+1)\boldsymbol{d}-r\}$*. Then we have* $\mathfrak{F}^t \in C^l((-2, 2) \times \mathcal{U}_{-k}, \mathcal{P}^{-h})$ *for a neighborhood of the origin* $\mathcal{U}_{-k} \subset \mathcal{P}^{-k}$*.*

(2) Let h *and* k *be like above with* $h \leq r - (l + 1)\boldsymbol{d}$*. Then given a function* $\mathcal{R}^{a,b}_{h,l}(\varrho, R)$*, we have* $\mathcal{R}^{a,b}_{h,l} \circ \mathfrak{F}^t = \mathcal{R}^{a,b}_{k,l}(t, \varrho, R)$ *and given a function* $\boldsymbol{S}^{a,b}_{h,l}(\varrho, R)$*, we have* $\boldsymbol{S}^{a,b}_{h,l} \circ \mathfrak{F}^t = \boldsymbol{S}^{a,b}_{k,l}(t, \varrho, R)$*.*

Proof. (3.38) follows by (3.29). By (3.30) we have

$$\mathbf{S} \in C^l((-2,2)\times \mathcal{U}_{-k}, \Sigma_{r-(l+1)\mathbf{d}}) \ , \quad q \text{ and } \mathfrak{F}^t_\varrho \in C^l((-2,2)\times \mathcal{U}_{-k}, \mathbb{R}^{n_0}).$$

By the above formulas we have $\mathfrak{F}^t_R \in C^l((-2,2)\times \mathcal{U}_{-k}, \Sigma_{r-(2l+1)\mathbf{d}} \cap \Sigma_{-k-l\mathbf{d}})$. This yields $\mathfrak{F}^t_R \in C^l((-2,2)\times \mathcal{U}_{-k}, \Sigma_{-h})$ and yields Claim (1).
By Claim (1), $\mathcal{R}^{a,b}_{h,l}\circ \mathfrak{F}^t \in C^l((-2,2)\times \mathcal{U}_{-k}, \mathbb{R}^{n_0})$. Let $(\varrho^t, R^t) = \mathfrak{F}^t(\varrho, R)$. Then

$$\begin{aligned} &|\mathcal{R}^{a,b}_{h,l}\circ \mathfrak{F}^t(\varrho, R)| = |\mathcal{R}^{a,b}_{h,l}(\varrho^t, R^t)| \le C' \|R^t\|^b_{\Sigma_{-h}} (\|R^t\|_{\Sigma_{-h}} + |\varrho^t|)^a \\ &\le C\|R\|^b_{\Sigma_{-h}} (\|R\|_{\Sigma_{-h}} + |\varrho|)^a \le C\|R\|^b_{\Sigma_{-k}} (\|R\|_{\Sigma_{-k}} + |\varrho|)^a, \end{aligned}$$

where the first inequality uses Definition (2.32), the second uses (3.34)–(3.35) for $s'' = -h$ and the last is obvious. Similarly by Claim (1), $\mathbf{S}^{a,b}_{h,l}\circ \mathfrak{F}^t \in C^l((-2,2)\times \mathcal{U}_{-k}, \Sigma_h) \subset C^l((-2,2)\times \mathcal{U}_{-k}, \Sigma_k)$ and

$$\begin{aligned} &\|\mathbf{S}^{a,b}_{h,l}(\varrho^t, R^t)\|_{\Sigma_k} \le \|\mathbf{S}^{a,b}_{h,l}(\varrho^t, R^t)\|_{\Sigma_h} \le C' \|R^t\|^b_{\Sigma_{-h}} (\|R^t\|_{\Sigma_{-h}} + |\varrho^t|)^a \\ &\le C\|R\|^b_{\Sigma_{-h}} (\|R\|_{\Sigma_{-h}} + |\varrho|)^a \le C\|R\|^b_{\Sigma_{-k}} (\|R\|_{\Sigma_{-k}} + |\varrho|)^a. \end{aligned}$$

□

To prove Theorem 6.4 we will need more information on $(\Pi(R(1)), R(1))$. This is provided by the following lemma.

Lemma 3.10. *Consider, for $\mathcal{D}$ the function in* (3.18) *at $\Pi = p_0$, the system*

$$\dot S(t) = \mathcal{D}(t, \Pi(R_0), S(t)) \ , \quad S(0) = R_0. \tag{3.39}$$

Then for $S' = S(1)$ and for $R' = R(1)$ with $R(t)$ the solution of (3.18) *with $R(0) = R_0$, we have (same indexes of Lemma 3.8)*

$$\begin{aligned} &\|R' - S'\|_{\Sigma_{-s'}} \le C\|R_0\|^{M_0+2}_{\Sigma_{-s}} \ , \\ &\Pi(R') - \Pi(S') = \mathcal{R}^{i,2M_0+1}_{s,l}(\Pi(R_0), R_0). \end{aligned} \tag{3.40}$$

Proof. Recall that for $\varrho = \Pi(R)$ we have $\dot\varrho = \langle R, \Diamond \mathcal{D}(t,\varrho,R)\rangle$. Similarly, for $\sigma = \Pi(S)$ we have $\dot\sigma = \langle S, \Diamond \mathcal{D}(t,\varrho_0,S)\rangle$, where $\varrho_0 = \Pi(R_0)$. So we have

$$\begin{aligned} \dot\varrho - \dot\sigma &= \langle R, \Diamond \mathcal{D}(t,\varrho,R)\rangle - \langle S, \Diamond \mathcal{D}(t,\varrho_0,S)\rangle \\ &= \langle R - S, \Diamond \mathcal{D}(t,\varrho,R)\rangle + \langle S, \Diamond (\mathcal{D}(t,\varrho_0,S) - \mathcal{D}(t,\varrho,R))\rangle. \end{aligned}$$

By (3.19) for fixed constants and using $s' \le r - \mathbf{d}$, we have

$$\begin{aligned} &|\dot\varrho - \dot\sigma| \lesssim \|R-S\|_{\Sigma_{-s'}} \|\mathcal{D}(t,\varrho,R)\|_{\Sigma_r} + \|S\|_{\Sigma_{-s'}} \|\mathcal{D}(t,\varrho_0,S) - \mathcal{D}(t,\varrho,R)\|_{\Sigma_r} \\ &\lesssim \|R-S\|_{\Sigma_{-s'}} \|R\|^{M_0}_{\Sigma_{-s}} (|\varrho| + \|R\|_{\Sigma_{-s'}})^i + |\varrho - \varrho_0| \ \|S\|_{\Sigma_{-s'}} \|(R,S)\|^{M_0}_{\Sigma_{-s'}} \\ &+ \|R-S\|_{\Sigma_{-s'}} \|S\|_{\Sigma_{-s'}} \|(R,S)\|^{M_0-1}_{\Sigma_{-s'}} (|(\varrho,\varrho_0)| + \|(R,S)\|_{\Sigma_{-s'}})^i. \end{aligned}$$

We have $\dot{R} - \dot{S} = \mathcal{D}(t, \varrho, R) - \mathcal{D}(t, \varrho_0, S) + J\mathcal{A}(t, \varrho, R)(t, \varrho, R) \cdot \Diamond R$ and hence for fixed constants we have, using $s \leq s' - \mathbf{d}$,

$$\begin{aligned}
&\|R - S\|_{\Sigma_{-s'}} \leq \int_0^t [\|\mathcal{D}(\varrho, R) - \mathcal{D}(\varrho_0, S)\|_{\Sigma_{-s'}} + |\mathcal{A}|\|R\|_{\Sigma_{-s}}]dt' \\
&\lesssim \int_0^t \big[\|R - S\|_{\Sigma_{-s'}} \|(R, S)\|^{M_0-1}_{\Sigma_{-s'}} (|(\varrho, \varrho_0)| + \|(R, S)\|_{\Sigma_{-s'}})^i \\
&+ |\varrho - \varrho_0| \ \|(R, S)\|^{M_0}_{\Sigma_{-s'}} + \|R\|^{M_0+2}_{\Sigma_{-s}}\big]dt'.
\end{aligned}$$

Recall that $|\varrho - \varrho_0| \leq C\|R_0\|^{M_0+1}_{\Sigma_{(l+1)\mathbf{d}-r}} (|\varrho_0| + \|R_0\|_{\Sigma_{(l+1)\mathbf{d}-r}})^i$ by (3.35), that $s < r - (l + 1)\mathbf{d}$ and that we have (3.34) for $s'' = -s, -s'$. Then by Gronwall inequality, the above inequalities yield

$$\begin{aligned}
&\|R(t) - S(t)\|_{\Sigma_{-s'}} \leq C\|R_0\|^{M_0+2}_{\Sigma_{-s}} \\
&|\varrho(t) - \sigma(t)| \leq C\|R_0\|^{2M_0+1}_{\Sigma_{-s}} (|\varrho_0| + \|R_0\|_{\Sigma_{-s}})^i.
\end{aligned} \tag{3.41}$$

This yields the bounds implicit in (3.40). The regularity follows from Lemma 3.8.

□

3.3 Darboux Theorem: end of the proof

Formally the proof should follow by $i_{\mathcal{X}^t}\Omega_t = -\alpha$, where $\Omega_t = (1 - t)\Omega_0 + t\Omega$, and by

$$\frac{d}{dt} (\mathfrak{F}_t^*\Omega_t) = \mathfrak{F}_t^* \left(L_{\mathcal{X}_t}\Omega_t + \frac{d}{dt}\Omega_t \right) = \mathfrak{F}_t^* \left(di_{\mathcal{X}^t}\Omega_t + d\alpha \right) = 0. \tag{3.42}$$

But while for [4, 7] the above formal computation falls within the classical framework of flows, fields and differential forms, in the case of [2, 6] this is not rigorous. In order to justify rigorously this computation, we will consider first a regularization of system (3.18).

Lemma 3.11. *Consider the system*

$$\begin{aligned}
&\dot{\tau}_j = T_j(t, \Pi, \Pi(R), R) \ , \quad \dot{\Pi}_j = 0 \ , \\
&\dot{R} = \mathcal{A}_j(t, \Pi, \Pi(R), R)J\langle\epsilon\Diamond\rangle^{-2}\Diamond_j R + \mathcal{D}_\epsilon(t, \Pi, \Pi(R), R),
\end{aligned} \tag{3.43}$$

where $\mathcal{D}_\epsilon = \mathcal{D} + \mathcal{A}_j P_{N_g(p_0)} J\Diamond_j (1 - \langle\epsilon\Diamond\rangle^{-2})R$.

(1) For $|\epsilon| \leq 1$ *system* (3.43) *satisfies all the conclusions of Lemma 3.43, if we replace* $\Diamond$ *in* (3.21) *with* $\langle\epsilon\Diamond\rangle^{-2}\Diamond$ *(resp.* $\mathcal{D}$ *in* (3.24) *with* $\mathcal{D}_\epsilon$*), with a fixed choice of constants* ε_1*,* ε_2*,* C*, and with a fixed choice of sets* $B_{\mathbb{R}^{n_0}}$*,* $B_{\Sigma_{-s}}$*.*

(2) For $\mathcal{X}^t$ the vector field of (3.18)*, denote by $\mathcal{X}^t_\epsilon$ the vector field of* (3.43)*. Let $n' > n + \boldsymbol{d}$ with $n, n' \in \mathbb{N}$. Then for $k \in \mathbb{Z} \cap [0, r]$ we have*

$$\lim_{\epsilon \to 0} \mathcal{X}^t_\epsilon = \mathcal{X}^t \text{ in } C^M((-3,3) \times \mathcal{U}^{n'}_{\varepsilon_0,k}, \widetilde{\mathcal{P}}^n) \text{ uniformly locally,} \tag{3.44}$$

that is uniformly on subsets of $(-3,3) \times \mathcal{U}^{n'}_{\varepsilon_0,k}$ bounded in $(-3,3) \times \widetilde{\mathcal{P}}^{n'}$.

(3) Denote by $\mathfrak{F}^t_\epsilon = (\mathfrak{F}^t_{\epsilon\tau}, \mathfrak{F}^t_{\epsilon R})$ the flow associated to (3.43) *at $\Pi = p_0$. Let s',s and k as in the statement of Lemma 3.8. Then there is a pair $0 < \varepsilon_1 < \varepsilon_0$ such that*

$$\lim_{\varepsilon \to 0} \mathfrak{F}^t_\epsilon = \mathfrak{F}^t \text{ in } C^{l-1}([-1,1] \times \mathcal{U}^{s'}_{\varepsilon_1,k}, \mathcal{U}^s_{\varepsilon_0,k}) \text{ uniformly locally.} \tag{3.45}$$

Proof. For claim (1), it is enough to check that $\mathcal{D}_\epsilon$ satisfies an estimate like the one of $\mathcal{D}$ in (3.23) for a fixed C for all $|\epsilon| \le 1$. Indeed, after this has been checked, the proof of Lemma 3.18 can be repeated verbatim, exploiting (A6) for $\epsilon \neq 0$ and with $\Diamond$ replaced by $\langle \epsilon \Diamond \rangle^{-2} \Diamond$.
The estimate on $\mathcal{D}_\epsilon$ needed for Claim (1) follows by the definition of $\mathcal{D}_\epsilon$, by the estimate on $\mathcal{D}$, by $P_{N_g(p_0)} = \mathbf{e}_a \langle \mathbf{e}^*_a, \ \rangle$ (sum on repeated indexes) for Schwartz functions $\mathbf{e}_a$ and $\mathbf{e}^*_a$ and, for $n \in \mathbb{N}$ with $n - 1 \ge s + \mathbf{d}$, and by

$$\begin{aligned} &\|P_{N_g(p_0)} J \Diamond_i (1 - \langle \epsilon \Diamond \rangle^{-2})\|_{B(\Sigma_{-r}, \Sigma_r)} \le \|\mathbf{e}_a \langle J \Diamond_i (1 - \langle \epsilon \Diamond \rangle^{-2}) \mathbf{e}^*_a, \ \rangle\|_{B(\Sigma_{-r}, \Sigma_r)} \\ &\le \|\mathbf{e}_a\|_{\Sigma_r} \|(1 - \langle \epsilon \Diamond \rangle^{-2}) \mathbf{e}^*_a\|_{\Sigma_{r+\mathbf{d}}} \le C(\epsilon) \|\mathbf{e}_a\|_{\Sigma_r} \ \|\mathbf{e}^*_a\|_{\Sigma_{r'}} \end{aligned} \tag{3.46}$$

$C(\epsilon) = \|\Diamond(1 - \langle \epsilon \Diamond \rangle^{-2})\|_{B(\Sigma_{r'}, \Sigma_{r+\mathbf{d}})}$ is bounded by (2.4) for $|\epsilon| \le 1$ for any pair (r', r) with $r' > r + \mathbf{d}$.
We consider now Claim (2). We have

$$\mathcal{X}^t - \mathcal{X}^t_\epsilon = \mathcal{A}_j(t, \varrho, R) \left(J(1 - \langle \epsilon \Diamond \rangle^{-2}) \Diamond_j R - P_{N_g(p_0)} J \Diamond_j (1 - \langle \epsilon \Diamond \rangle^{-2}) R \right).$$

We have $P_{N_g(p_0)} J \Diamond_j (1 - \langle \epsilon \Diamond \rangle^{-2}) R \overset{\epsilon \to 0}{\to} 0$ for $R \in \Sigma_{n'}$ for any $n' \in \mathbb{Z}$ because in fact $C(\epsilon) \overset{\epsilon \to 0}{\to} 0$ by (2.5), with $C(\epsilon)$ defined like above for any pair (r', r) with $r' > r + \mathbf{d}$.
Still by (2.5), for $n > n' + \mathbf{d}$ and for $R \in \Sigma_{n'}$ we have by (A5)

$$\begin{aligned} &\|J \Diamond (1 - \langle \epsilon \Diamond \rangle^{-2}) R\|_{\Sigma_n} \le \|\Diamond (1 - \langle \epsilon \Diamond \rangle^{-2})\|_{B(\Sigma_{n'}, \Sigma_n)} \|R\|_{\Sigma_{n'}} \\ &\le C \|(1 - \langle \epsilon \Diamond \rangle^{-2})\|_{B(\Sigma_{n'}, \Sigma_{n+\mathbf{d}})} \|R\|_{\Sigma_{n'}} \overset{\epsilon \to 0}{\to} 0. \end{aligned} \tag{3.47}$$

These facts yield (3.44).
We turn now to Claim (3) and to (3.45). By the Rellich criterion, the embedding $\Sigma_a \hookrightarrow \Sigma_b$ for $a > b$ is compact. Hence also $\mathcal{P}^a \hookrightarrow \mathcal{P}^b$ is compact. Then (3.45) follows by the Ascoli–Arzela Theorem by a standard argument.

□

Corollary 3.12. *Consider* (3.18) *defined by the field* $\mathcal{X}^t$ *and consider indexes and notation of Lemma 3.8 (in particular we have* $M_0 = 1$ *and* $i = 1$ *in* (3.19) *and elsewhere;* r *and* M *can be arbitrary). Consider* s',s *and* k *as in 3.8. Then for the map* $\mathfrak{F}^t \in C^l(\mathcal{U}^{s'}_{\varepsilon_1,k}, \widetilde{\mathcal{P}}^s)$ *derived from* (3.25)*, we have* $\mathfrak{F}^{1*}\Omega = \Omega_0$.

Proof. Ω_0 is constant in the coordinate system (τ, Π, R) where $R \in N_g^{\perp}(\mathcal{H}^*_{p_0})$, with $\Omega_0 = d\tau_j \wedge d\Pi_j + \langle J^{-1}\ ,\ \rangle$, where we apply $\langle J^{-1}\ ,\ \rangle$ only to vectors in the R space. Hence Ω_0 is C^∞ in $R \in L^2$, τ and Π, with values in $B^2(L^2, \mathbb{R})$. From Lemma 3.3 we have that $d\alpha$, so also Ω by $\Omega = \Omega_0 + d\alpha$, belongs to $C^\infty(\mathcal{U}^s_{\varepsilon_0,k}, B^2(\widetilde{\mathcal{P}}, \mathbb{R}))$ for an $\epsilon_0 > 0$, and so also to $C^\infty(\mathcal{U}^s_{\varepsilon_0,k}, B^2(\widetilde{\mathcal{P}}^s, \mathbb{R}))$. Let now $r - (l+1)\mathbf{d} \geq s' \geq s + l\mathbf{d}$ and $k \in \mathbb{Z} \cap [0, r - (l+1)\mathbf{d}]$. Then for a fixed $0 < \varepsilon_2 \ll \varepsilon_1$ and for all $|\epsilon| \leq 1$ we have

$$\mathfrak{F}^t_\epsilon \in C^l((-2,2) \times \mathcal{U}^{s'}_{\varepsilon_2,k}, \mathcal{U}^s_{\varepsilon_1,k}), \quad \mathfrak{F}^t_\epsilon(\mathcal{U}^{s'}_{\varepsilon_2,k}) \subset \mathcal{U}^{s'}_{\varepsilon_1,k} \text{ for all } |t| \leq 2 \tag{3.48}$$

by Lemma 3.8, for a fixed $l \geq 2$. By Lemma 3.11 we have uniformly locally

$$\lim_{\varepsilon \to 0} \mathfrak{F}^t_\epsilon = \mathfrak{F}^t \text{ in } C^l([-1,1] \times \mathcal{U}^{s'}_{\varepsilon_2,k}, \mathcal{U}^s_{\varepsilon_1,k}). \tag{3.49}$$

Let us take $0 < \varepsilon_3 \ll \varepsilon_2$ s.t. $\mathfrak{F}^t_\epsilon(\mathcal{U}^{s'}_{\varepsilon_3,k}) \subset \mathcal{U}^{s'}_{\varepsilon_2,k}$ for all $|t| \leq 2$ and $|\epsilon| \leq 1$.
In $\mathcal{U}^{s'}_{\varepsilon_3,k}$ the following computation is valid because $\mathcal{X}^t_\epsilon$ is a standard vector field in $\mathcal{U}^{s'}_{\varepsilon_1,k}$ and similarly Ω_t is a regular differential form therein:

$$\begin{aligned}
&\mathfrak{F}^{1*}_\epsilon \Omega - \Omega_0 = \int_0^1 \frac{d}{dt}\left(\mathfrak{F}^{t*}_\epsilon \Omega_t\right) dt = \int_0^1 \mathfrak{F}^{t*}_\epsilon \left(L_{\mathcal{X}^t_\epsilon} \Omega_t + \frac{d}{dt}\Omega_t \right) dt \\
&= d \int_0^1 \mathfrak{F}^{t*}_\epsilon \left(i_{\mathcal{X}^t_\epsilon} \Omega_t + \alpha \right) dt,
\end{aligned}$$

where we recall $\Omega_t = \Omega_0 + t(\Omega - \Omega_0)$.
If we consider a ball $\mathbf{B}$ in $\mathcal{U}^{s'}_{\varepsilon_3,k}$, in the notation of Lemma 3.1, for some function $\psi_\epsilon \in C^1(\mathbf{B}, \mathbb{R})$ we can write

$$\mathfrak{F}^{1*}_\epsilon (B_0 + \alpha) - B_0 + d\psi_\epsilon = \int_0^1 \mathfrak{F}^{t*}_\epsilon \left(i_{\mathcal{X}^t_\epsilon} \Omega_t + \alpha \right) dt, \tag{3.50}$$

By (3.48)–(3.49) we have

$$\lim_{\epsilon \to 0} (\mathfrak{F}^{1*}_\epsilon (B_0 + \alpha) - B_0) = \mathfrak{F}^{1*}(B_0 + \alpha) - B_0 \text{ in } C^{l-1}(\mathcal{U}^{s'}_{\varepsilon_3,k}, B(\widetilde{\mathcal{P}}^{s'}, \mathbb{R})).$$

The set $\Gamma := \{\mathfrak{F}^t_\epsilon(\mathbf{B}) : |t| \leq 2, |\epsilon| \leq 1\}$ is a bounded subset in $\mathcal{U}^{s'}_{\varepsilon_2,k}$ because of (3.34)–(3.35). Then we have

$$\lim_{\epsilon \to 0} \mathcal{X}^t_\epsilon = \mathcal{X}^t \text{ in } C^0((-2,2) \times \Gamma, \widetilde{\mathcal{P}}^s) \text{ uniformly .}$$

Hence by $i_{\mathcal{X}^t}\Omega_t = -\alpha$ we get

$$\lim_{\epsilon \to 0} \left(i_{\mathcal{X}^t_\epsilon} \Omega_t + \alpha \right) = i_{\mathcal{X}^t} \Omega_t + \alpha = 0 \text{ in } C^0((-2,2) \times \Gamma, B(\widetilde{\mathcal{P}}^s, \mathbb{R})) \text{ uniformly.}$$

This implies

$$\lim_{\epsilon\to 0}\|\int_0^1 \mathfrak{F}_\epsilon^{t*}\left(i_{\mathcal{X}_\epsilon^t}\Omega_t+\alpha\right)dt\|_{L^\infty(\mathbf{B},B(\widetilde{\mathcal{P}}^{s'},\mathbb{R}))}$$
$$\le C\lim_{\epsilon\to 0}\|i_{\mathcal{X}_\epsilon^t}\Omega_t+\alpha\|_{L^\infty([0,1]\times\Gamma,B(\widetilde{\mathcal{P}}^{s},\mathbb{R}))}=0,$$

for C an upper bound to the norms $\|(\mathfrak{F}_\epsilon^{t*})_{|\mathfrak{F}_\epsilon^t(\upsilon)}:B(\widetilde{\mathcal{P}}^{s'},\mathbb{R})\to B(\widetilde{\mathcal{P}}^{s},\mathbb{R})\|$ as υ varies in $\mathbf{B}$. Notice that $C<\infty$ by (3.45).
By (3.50) we conclude that uniformly

$$\lim_{\epsilon\to 0}d\psi_\epsilon = B_0-\mathfrak{F}^{1*}(B_0+\alpha)\text{ in }C^0(\mathbf{B},B(\widetilde{\mathcal{P}}^{s'},\mathbb{R})).$$

Normalizing $\psi_\epsilon(\upsilon_0)=0$ at some given $\upsilon_0\in\mathbf{B}$, it follows that also ψ_ϵ converges locally uniformly to a function ψ_0 with $d\psi_0=B_0-\mathfrak{F}^{1*}(B_0+\alpha)$. Taking the exterior differential, we conclude that $\mathfrak{F}^{1*}\Omega=\Omega_0$ in $C^\infty(\mathcal{U}^{s'}_{\varepsilon_3,k},B^2(\widetilde{\mathcal{P}}^{s'},\mathbb{R}))$.
□

4 Pullback of the Hamiltonian

In the somewhat abstract set up of this paper it is particularly important to have a general description of the pullbacks of the Hamiltonian K. Our main goal in this section is formula (4.14). This formula and its related expansion in Lemma 5.4 obtained splitting R in discrete and continuous modes, play a key role in the Birkhoff normal forms argument.

The first and quite general result is the following consequence of Lemma 3.8.

Lemma 4.1. *Consider $\mathfrak{F}=\mathfrak{F}_1\circ\cdots\circ\mathfrak{F}_L$ with $\mathfrak{F}_j=\mathfrak{F}_j^{t=1}$ transformations as of Lemma 3.8. Suppose that for j we have $M_0=m_j$, with given numbers $1\le m_1\le ...\le m_L$. Suppose also that all the j we have the same pair r and M, which we assume sufficiently large. Let $i_j=1$ if $m_j=1$. Fix $0<m'<M$*

(1) Let $r>2L(m'+1)\mathbf{d}+s'_L>4L(m'+1)\mathbf{d}+s_1$, $s_1\ge\mathbf{d}$. Then, for any $\varepsilon>0$ there exists a $\delta>0$ such that $\mathfrak{F}\in C^{m'}(\mathcal{U}^{s'_L}_{\delta,a},\mathcal{U}^{s_1}_{\varepsilon,h})$ for $0\le a\le h$ and $0\le h<r-(m'+1)\mathbf{d}$.

(2) Let $r>2L(m'+1))\mathbf{d}+h>4L(m'+1)\mathbf{d}+a$, $a\ge 0$. The above composition, interpreting the $\mathfrak{F}_j$'s as maps in the (ϱ,R) variables as in Lemma 3.9, yields also $\mathfrak{F}\in C^{m'}(\mathcal{U}_{-a},\mathcal{P}^{-h})$ for $\mathcal{U}_{-a}$ a sufficiently small neighborhood of the origin in $\mathcal{P}^{-a}$.

(3) For $\mathcal{U}_{-a}\subset\mathcal{P}^{-a}$ like above and for functions $\mathcal{R}^{i,j}_{a,m'}\in C^{m'}(\mathcal{U}_{-a},\mathbb{R})$ and $\mathbf{S}^{i,j}_{a,m'}\in C^{m'}(\mathcal{U}_{-a},\Sigma_a)$, the following formulas hold:

$$\begin{aligned}&\Pi(R'):=\Pi(R)\circ\mathfrak{F}=\Pi(R)+\mathcal{R}^{i_1,m_1+1}_{a,m'}(\Pi(R),R),\\&p':=p\circ\mathfrak{F}=p+\mathcal{R}^{i_1,m_1+1}_{a,m'}(\Pi(R),R),\\&\Phi_{p'}=\Phi_p+\mathbf{S}^{i_1,m_1+1}_{a,m'}(\Pi(R),R).\end{aligned}\tag{4.1}$$

(4) For a function F such that $F(e^{J\tau\cdot\Diamond}U) \equiv F(U)$ we have

$$F\circ\mathfrak{F}(U) = F\left(\Phi_p + P(p)(R+\mathbf{S}^{i_1,m_1}_{k',m'}) + \boldsymbol{S}^{i_1,m_1+1}_{k',m'}\right),\ k' = r-7L(m'+1)\mathbf{d}.$$

Proof. Recall that by (3.25) we have $\mathfrak{F}_j \in C^{m'}(\mathcal{U}^{s'_j}_{\varepsilon'_j,h}, \mathcal{U}^{s_j}_{\varepsilon_j,h})$ for $r-(m'+1)\mathbf{d} > s'_j \geq s_j + m'\mathbf{d}$ and appropriate choice of the $0<\varepsilon'_j<\varepsilon_j$ and for $h\in\mathbb{Z}\cap[0, r-(m'+1)\mathbf{d}]$. So for the composition we have $\mathfrak{F}\in C^{m'}(\mathcal{U}^{\kappa}_{\varepsilon'_L,a}, \mathcal{U}^{s_1}_{\varepsilon_1,h})$ for $a\leq h$. The inequalities $r > 2L(m'+1)\mathbf{d}+s'_L > 4L(m'+1)\mathbf{d}+s_1$, $s_1 \geq \ \mathbf{d}$ can be accommodated since r is assumed sufficiently large. This yields claim (1).
By Lemma 3.9 we have $\mathfrak{F}_j\in C^{m'}(\mathcal{U}_{-h+jm'\mathbf{d}}, \mathcal{P}^{-h+(j-1)m'\mathbf{d}})$ with $\mathcal{U}_{-h+jm'\mathbf{d}}\subset\mathcal{P}^{-h+jm'\mathbf{d}}$ a neighborhood of the origin. So for the composition we have $\mathfrak{F}\in C^{m'}(\mathcal{U}_{-a}, \mathcal{P}^{-h})$ for $a\leq h-Lm'\mathbf{d}$. The conditions $r>2L(m'+1)\mathbf{d}+h$, $h>4L(m'+1)\mathbf{d}+a$ and $a\geq 0$, can be accommodated since r is assumed sufficiently large. This yields claim (2).
We now prove (4.1). Let first $L=1$. By (3.21) we have $R' := (\mathfrak{F}_1)_R(\Pi(R),R) = e^{Jq_1\cdot\Diamond}(R+\mathbf{S}^{i_1,m_1}_{r-(m'+1)\mathbf{d},m'})$, where we use $M>m'$. Here we will omit the variables $(\Pi(R),R)$ in the $\mathbf{S}$'s and $\mathcal{R}$'s. Then we have for $a' = r-(m'+1)\mathbf{d}$

$$\Pi(R') = \Pi(R+\mathbf{S}^{i_1,m_1}_{a',m'}) = \Pi(R) + \mathcal{R}^{i_1,m_1+1}_{a'-\mathbf{d},m'}. \tag{4.2}$$

Here we have used

$$|\langle R, \Diamond\mathbf{S}^{i_1,m_1}_{a',m'}\rangle| \leq \|R\|_{\Sigma_{-a'+\mathbf{d}}}\|\mathbf{S}^{i_1,m_1}_{a',m'}\|_{\Sigma_{a'}}.$$

By $p_j = \Pi_j - \Pi_j(R) + \mathcal{R}^{0,2}(\Pi(R),R)$ we get

$$\begin{aligned} p'_j &= \Pi_j - \Pi_j(R') + \mathcal{R}^{0,2}(\Pi(R'),R') \\ &= \Pi_j - \Pi_j(R) + \mathcal{R}^{0,2}(\Pi(R),R) + \mathcal{R}^{i_1,m_1+1}_{a'-\mathbf{d},m'} = p_j + \mathcal{R}^{i_1,m_1+1}_{a'-\mathbf{d},m'}. \end{aligned} \tag{4.3}$$

This yields (4.1) for $L=1$ since $a\leq r-4(m'+1)\mathbf{d} < a'-\mathbf{d}$. We extend the proof to the case $L>1$. We write here and below $\mathfrak{F}' := \mathfrak{F}_1\circ\cdots\circ\mathfrak{F}_{L-1}$. We suppose that $\mathfrak{F}'_R(\Pi(R),R) = e^{Jq\cdot\Diamond}(R+\mathbf{S}^{i_1,m_1}_{a'_{L-1},m'})$ for $a'_{L-1}\leq r-2(L-1)m'\mathbf{d}$, which is true for $L-1=1$. Then

$$\begin{aligned} R' &= e^{J(q\circ\mathfrak{F}_L)\cdot\Diamond}\left(e^{Jq_L\cdot\Diamond}(R+\mathbf{S}^{i_L,m_L}_{r-(m'+1)\mathbf{d},m'}) + \mathbf{S}^{i_1,m_1}_{a'_{L-1},m'}\circ\mathfrak{F}_L\right) \\ &= e^{J(q\circ\mathfrak{F}_L+q_L)\cdot\Diamond}\left(R+\mathbf{S}^{i_L,m_L}_{r-(m'+1)\mathbf{d},m'}) + e^{-Jq_L\cdot\Diamond}\mathbf{S}^{i_1,m_1}_{a'_{L-1}-m'\mathbf{d},m'}\right), \end{aligned}$$

where $q_L = \mathcal{R}^{0,m_L+1}_{r-(m'+1)\mathbf{d},m'}$ and where we used the last claim in Lemma 3.9. Since $e^{-Jq_L\cdot\Diamond}\mathbf{S}^{i_1,m_1}_{a'_{L-1}-m'\mathbf{d},m'} = \mathbf{S}^{i_1,m_1}_{a'_{L-1}-2m'\mathbf{d},m'}$ we conclude that there is an expansion $R' = e^{Jq\cdot\Diamond}(R+\mathbf{S}^{i_1,m_1}_{a'_L,m'})$ for $a'_L\leq a'_{L-1}-2m'\mathbf{d}$. Then

$$\mathfrak{F}_R(\Pi(R),R) = e^{Jq\cdot\Diamond}(R+\mathbf{S}^{i_1,m_1}_{a'_L,m'}),\quad a'_L := r-2Lm'\mathbf{d}. \tag{4.4}$$

For $a' = a'_L$ formulas (4.2)–(4.3) continue to hold. By $a < a'_L - \mathbf{d}$ this yields (4.1).
We consider the last statement of Lemma 3.21. For $a' = r - (m'+1)\mathbf{d}$ we have

$$\begin{aligned}
&F(\mathfrak{F}_1(U)) = F(\Phi_{p'} + P(p')e^{Jq_1\cdot\Diamond}(R + \mathbf{S}^{i_1,m_1}_{a',m'})) = \\
&F(\Phi_p + P(p)e^{Jq_1\cdot\Diamond}(R + \mathbf{S}^{i_1,m_1}_{a',m'}) + \mathbf{S}^{i_1,m_1+1}_{a'+\mathbf{d},m'}) = \\
&F\left(e^{Jq_1\cdot\Diamond}\left(\Phi_p + P(p)(R + \mathbf{S}^{i_1,m_1}_{a',m'}) + Y\right)\right)
\end{aligned}$$

with

$$Y = (e^{Jq_1\cdot\Diamond} - 1)\Phi_p + [P(p), e^{Jq_1\cdot\Diamond}](R + \mathbf{S}^{i_1,m_1}_{a',m'}) + e^{-Jq_1\cdot\Diamond}\mathbf{S}^{i_1,m_1+1}_{a'-\mathbf{d},m'}.$$

We claim

$$Y = \mathbf{S}^{i_1,m_1+1}_{a'-2m'\mathbf{d},m'}. \tag{4.5}$$

To prove (4.5) we use $(e^{Jq_1\cdot\Diamond} - 1)\Phi_p = \mathbf{S}^{i_1,m_1+1}_{r-(m'+1)\mathbf{d},m'} = \mathbf{S}^{i_1,m_1+1}_{a',m'}$. This follows from $\Phi_p \in C^\infty(\mathcal{O}, \mathcal{S})$ and

$$\left|(e^{Jq_1\cdot\Diamond} - 1)\Phi_p\right|_{\Sigma_l} \le |q_{1j}| \int_0^1 \left|e^{tJq_1\cdot\Diamond}\Diamond_j\Phi_p\right|_{\Sigma_l} dt \le C_l|q_{1j}|\,|\Diamond_j\Phi_p|_{\Sigma_l}. \tag{4.6}$$

Schematically we have, summing over repeated indexes and for $\mathbf{e}_j, \mathbf{e}_j^* \in \mathcal{S}$,

$$\begin{aligned}
&[P(p), e^{Jq_1\cdot\Diamond}] = [e^{Jq_1\cdot\Diamond}, P_{N_g}(p)] = e^{Jq_1\cdot\Diamond}\mathbf{e}_j\langle\mathbf{e}_j^*, \ \rangle - \mathbf{e}_j\langle e^{-Jq_1\cdot\Diamond}\mathbf{e}_j^*, \ \rangle \\
&= (e^{Jq_1\cdot\Diamond} - 1)\mathbf{e}_j\langle\mathbf{e}_j^*, \ \rangle - \mathbf{e}_j\langle(e^{-Jq_1\cdot\Diamond} - 1)\mathbf{e}_j^*, \ \rangle \\
&= \mathbf{S}^{0,m_1+1}_{r-(m'+1)\mathbf{d},m'}\langle\mathbf{e}_j^*, \ \rangle + \mathbf{e}_j\langle\mathbf{S}^{0,m_1+1}_{r-(m'+1)\mathbf{d},m'}, \ \rangle.
\end{aligned}$$

This yields for any $a'' \le a' = r - (m'+1)\mathbf{d}$

$$[P(p), e^{Jq_1\cdot\Diamond}](R + \mathbf{S}^{i_1,m_1}_{a'',m'}) = \mathbf{S}^{i_1,m_1+2}_{a'',m'}.$$

We have $e^{-Jq_1\cdot\Diamond}\mathbf{S}^{i_1,m_1+1}_{a'-\mathbf{d},m'} = \mathbf{S}^{i_1,m_1+1}_{a'-(m'+1)\mathbf{d},m'}$. Then (4.5) is proved. Then

$$F(\mathfrak{F}_1(U)) = F\left(\Phi_p + P(p)(R + \mathbf{S}^{i_1,m_1}_{a'-2m'\mathbf{d},m'}) + \mathbf{S}^{i_1,m_1+1}_{a'-2m'\mathbf{d},m'}\right) \tag{4.7}$$

for $a' = r - (m'+1)\mathbf{d}$. This proves the last sentence of our lemma for $L = 1$.
For $L > 1$ set once more $\mathfrak{F}' := \mathfrak{F}_1 \circ \cdots \circ \mathfrak{F}_{L-1}$. We assume by induction that $F(\mathfrak{F}'(U))$ equals the rhs of (4.7) for $a' = a'_{L-1} := r - 2(L-1)m'\mathbf{d}$. Then using $\mathbf{S}^{i_1,m_1}_{l,m'} \circ \mathfrak{F}_L = \mathbf{S}^{i_1,m_1}_{l-m'\mathbf{d},m'}$ from Lemma 3.9, by (4.1) for $\mathfrak{F} = \mathfrak{F}_L$ and by (4.5) with the index 1 replaced by index L, we get

$$\begin{aligned}
&F(\mathfrak{F}(U)) = F\big(\Phi_{p'} + \\
&P(p')e^{Jq_L\cdot\Diamond}(R + \mathbf{S}^{i_L,m_L}_{r-(m'+1)\mathbf{d},m'}) + P(p')\mathbf{S}^{i_1,m_1}_{a'_{L-1}-m'\mathbf{d},m'} + \mathbf{S}^{i_1,m_1+1}_{a'_{L-1}-m'\mathbf{d},m'}\big) \\
&= F\left(e^{Jq_L\cdot\Diamond}\left[\Phi_p + P(p)(R + \mathbf{S}^{i_1,m_1}_{a'_{L-1}-m'\mathbf{d},m'}) + \mathbf{S}^{i_1,m_1+1}_{a'_{L-1}-2m'\mathbf{d},m'}\right]\right).
\end{aligned}$$

We conclude that $F(\mathfrak{F}(U))$ equals the rhs of (4.7) for $a'_L = r - 2Lm'\mathbf{d}$. In particular this proves the last sentence of our lemma for any L.

$\square$

Lemma 4.2. *For fixed vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *and for* B *sufficiently regular with* $B(0) = 0$*, we have*

$$\begin{aligned} &B(|\mathbf{u}+\mathbf{v}|_1^2) = B\left(|\mathbf{u}|_1^2\right) + B(|\mathbf{v}|_1^2) \\ &+ \sum_{j=0}^{3} \int_{[0,1]^2} \frac{t^j}{j!} (\partial_t^{j+1})_{|t=0} \partial_s [B(|s\mathbf{u}+t\mathbf{v}|_1^2)] \ dtds \\ &+ \int_{[0,1]^2} dtds \int_0^t \partial_\tau^5 \partial_s [B(|s\mathbf{u}+\tau\mathbf{v}|_1^2)] \frac{(t-\tau)^3}{3!} \ d\tau. \end{aligned} \tag{4.8}$$

Proof. Follows by Taylor expansion in t of

$$\begin{aligned} &B(|\mathbf{u}+\mathbf{v}|_1^2) = B\left(|\mathbf{u}|_1^2\right) + \int_0^1 \partial_t [B(|\mathbf{u}+t\mathbf{v}|_1^2)] dt = \\ &B\left(|\mathbf{u}|_1^2\right) + B(|\mathbf{v}|_1^2) + \int_{[0,1]^2} dtds \ \partial_s \partial_t [B(|s\mathbf{u}+t\mathbf{v}|_1^2)]. \end{aligned}$$

$\square$

Lemma 4.3. *Consider a transformation* $\mathfrak{F} = \mathfrak{F}_1 \circ \cdots \circ \mathfrak{F}_L$ *like in Lemma 4.1 and with* $m_1 = 1$*, with same notations, hypotheses and conclusions. In particular we suppose* r *and* M *sufficiently large that the conclusions of Lemma 4.1 hold for preassigned sufficiently large* $s = s'_L$*,* k' *and* m'*. Let* $k \le k' - \max\{\boldsymbol{d}, ord(\mathcal{D})\}$ *and* $m \le m'$*. Then there are a* $\underline{\psi}(\varrho) \in C^\infty$ *with* $\underline{\psi}(\varrho) = O(|\varrho|^2)$ *near 0 and a small* $\varepsilon > 0$ *such that in* $\mathcal{U}^s_{\varepsilon,k}$ *we have the expansion*

$$K \circ \mathfrak{F} = \underline{\psi}(\Pi(R)) + \frac{1}{2}\Omega(\mathcal{H}_p P(p)R, P(p)R) + \mathcal{R}^{1,2}_{k,m} + E_P(P(p)R) + \boldsymbol{R}'' \tag{4.9}$$

$$\boldsymbol{R}'' := \sum_{d=2}^{4} \langle B_d(R, \Pi(R)), (P(p)R)^d \rangle + \int_{\mathbb{R}^3} B_5(x, R, R(x), \Pi(R))(P(p)R)^5(x) dx$$

with:

- $\mathcal{R}^{1,2}_{k,m} = \mathcal{R}^{1,2}_{k,m}(\Pi(R), R)$*;*
- $B_2(0,0) = 0$*;*
- $(P(p)R)^d(x)$ *represent* $d-$*products of components of* $P(p)R$*;*
- $B_d(\cdot, R, \varrho) \in C^m(\mathcal{U}_{-k}, \Sigma_k(\mathbb{R}^3, B((\mathbb{R}^{2N})^{\otimes d}, \mathbb{R})))$ *for* $2 \le d \le 4$ *with* $\mathcal{U}_{-k} \subset \mathcal{P}^{-k}$ *a neighborhood of the origin;*
- *for* $\zeta \in \mathbb{R}^{2N}$ *with* $|\zeta| \le \varepsilon$ *and* $(\varrho, R) \in \mathcal{U}_{-k}$ *we have for* $i \le m$

$$\|\nabla^i_{R,\zeta,\varrho} B_5(R, \zeta, \varrho)\|_{\Sigma_k(\mathbb{R}^3, B((\mathbb{R}^{2N})^{\otimes 5}, \mathbb{R})} \le C_i. \tag{4.10}$$

Proof. Here we will omit the variables $(\Pi(R), R)$ in the $\mathbf{S}$'s and $\mathcal{R}$'s.
By Lemma 4.1 for $m \le m' \le M$, $k + \max\{\mathbf{d}, \mathrm{ord}(\mathcal{D})\} \le k' \le r - L(m'+2)\mathbf{d}$, we have

$$\begin{aligned} &K(\mathfrak{F}(U)) = E(\Phi_p + P(p)R + P(p)\mathbf{S}^{1,1}_{k',m'} + \mathbf{S}^{1,2}_{k',m'}) - E(\Phi_{p_0}) \\ &- (\lambda_j(p) + \mathcal{R}^{1,2}_{k,m})\left(\Pi_j(\Phi_p + P(p)R) + \mathcal{R}^{1,2}_{k,m} - \Pi_j(\Phi_{p_0})\right), \end{aligned} \tag{4.11}$$

where, by (4.1), we have used $p' := p \circ \mathfrak{F} = p + \mathcal{R}^{1,2}_{k,m}$ and where by $k \le k' - \mathbf{d}$

$$\Pi_j(\Phi_p + P(p)R + P(p)\mathbf{S}^{1,1}_{k',m'} + \mathbf{S}^{1,2}_{k',m'}) = \Pi_j(\Phi_p + P(p)R) + \mathcal{R}^{1,2}_{k,m}.$$

Set now $\Psi = \Phi_p + P(p)\mathbf{S}^{1,1}_{k',m'} + \mathbf{S}^{1,2}_{k',m'}$. By (4.8) for $\mathbf{u} = \Psi$ and $\mathbf{v} = P(p)R$

$$\begin{aligned} &E_P(\Psi + P(p)R) = E_P(\Psi) + E_P(P(p)R) \\ &+ \sum_{j=0}^{1} \int_{\mathbb{R}^3} dx \int_{[0,1]^2} \frac{t^j}{j!}(\partial_t^{j+1})_{|t=0}\partial_s[B(|s\Psi + tP(p)R|_1^2)]dtds \\ &+ \sum_{j=2}^{3} \int_{\mathbb{R}^3} dx \int_{[0,1]^2} \frac{t^j}{j!}(\partial_t^{j+1})_{|t=0}\partial_s[B(|s\Psi + tP(p)R|_1^2)]dtds \\ &+ \int_{\mathbb{R}^3} dx \int_{[0,1]^2} dtds \int_0^t \partial_\tau^5 \partial_s[B(|s\Psi + \tau P(p)R|_1^2)]\frac{(t-\tau)^3}{3!}d\tau. \end{aligned} \tag{4.12}$$

The last two lines can be incorporated in $\mathbf{R}''$. For example, schematically we have

$$\partial_\tau^5 \partial_s B(|s\Phi_p + \tau P(p)R|_1^2) \sim \widetilde{B}(s\Phi_p + \tau P(p)R)\ \Phi_p\ (P(p)R)^5,$$

for some $\widetilde{B}(Y) \in C^\infty(\mathbb{R}^{2N}, B^6(\mathbb{R}^{2N}, \mathbb{R}))$. This produces a term which can be absorbed in the B_5 term of $\mathbf{R}''$. In particular, (4.10) follows from (2.2). The terms in the third line of (4.12) can be treated similarly yielding terms which end in the B_d term of $\mathbf{R}''$ with $d = j + 1$.
The second line of (4.12) equals

$$\begin{aligned} &\int_{\mathbb{R}^3} dx \int_{[0,1]^2} dtds \sum_{j=0}^{1} \frac{t^j}{j!}(\partial_t^{j+1})_{|t=0}\partial_s \Big\{ \ B(|s\Phi_p + tP(p)R|_1^2) + \\ &+ \int_0^1 d\tau \partial_\tau [B(|s(\Phi_p + \tau(P(p)\mathbf{S}^{1,1}_{k',m'} + \mathbf{S}^{1,2}_{k',m'}) + tP(p)R|_1^2)] \ \Big\}. \end{aligned} \tag{4.13}$$

The contribution from the last line of (4.13) can be incorporated in $\mathbf{R}'' + \mathcal{R}^{1,2}_{k,m}$.
By $k \le k' - \mathrm{ord}(\mathcal{D})$ we have

$$\begin{aligned} E_K(\Psi + P(p)R) &= E_K(\Psi) + \langle \mathcal{D}\Phi_p, P(p)R\rangle \\ &+ \overbrace{\langle \mathcal{D}(P(p)\mathbf{S}^{1,1}_{k',m'} + \mathbf{S}^{1,2}_{k',m'}), P(p)R\rangle}^{\mathcal{R}^{1,2}_{k,m}} + E_K(P(p)R). \end{aligned}$$

Notice that from the $j = 0$ term in the first line of (4.13) we get

$$2\int_{\mathbb{R}^3} dx \int_0^1 ds\partial_s[B'(|s\Phi_p|_1^2)s\Phi_p \cdot_1 P(p)R] = 2\int_{\mathbb{R}^3} dx B'(|\Phi_p|_1^2)\Phi_p \cdot_1 P(p)R$$
$$= \langle \nabla E_P(\Phi_p), P(p)R\rangle.$$

By (2.6) and (2.16), that is $\nabla E(\Phi_p) = \lambda(p)\cdot \Diamond\Phi_p \in N_g(\mathcal{H}_p^*)$, and by $P(p)R \in N_g^{\perp}(\mathcal{H}_p)$, we have

$$\langle \mathcal{D}\Phi_p, P(p)R\rangle + \langle \nabla E_P(\Phi_p), P(p)R\rangle = \langle \nabla E(\Phi_p), P(p)R\rangle = 0.$$

The $j = 1$ term in the first line of (4.13) is $\frac{1}{2}\langle \nabla^2 E_P(\Phi_p)P(p)R, P(p)R\rangle$ which summed to the $E_K(P(p)R)$ in (4) yields the $\frac{1}{2}\Omega(\mathcal{H}_p P(p)R, P(p)R)$ in (4.9). We have $E_K(\Psi) + E_P(\Psi) = E(\Psi)$ and

$$E(\Psi) = E(\Phi_p) + \overbrace{\langle \nabla E(\Phi_p), P(p)\mathbf{S}^{1,1}_{k',m'}\rangle}^{0} + \overbrace{\langle \nabla E(\Phi_p), \mathbf{S}^{1,2}_{k',m'}\rangle}^{\mathcal{R}^{1,2}_{k,m}} + \mathcal{R}^{1,2}_{k,m}.$$

The last term we need to analyze, for $d(p) := E(\Phi_p) - \lambda(p)\cdot \Pi(\Phi_p)$, is

$$E(\Phi_p) - E(\Phi_{p_0}) - \sum_j \lambda_j(p)(\Pi_j(\Phi_p) - \Pi_j(\Phi_{p_0}))$$
$$= d(p) - d(p_0) - \sum_j (\lambda_j(p_0) - \lambda_j(p))p_{0j} =: \widetilde{\psi}(p, p_0),$$

where $\widetilde{\psi}(p, p_0) = O((p - p_0)^2)$ by $\partial_{p_j} d(p) = -p\cdot \partial_{p_j}\lambda(p)$. Notice that $\widetilde{\psi} \in C^\infty(\mathcal{O}^2, \mathbb{R})$. Now recall that in the initial system of coordinates we have $p' = \Pi - \Pi(R') + \mathcal{R}^{0,2}(\Pi(R'), R')$. Substituting p' and $\Pi(R')$ by means of (4.1), and R' by means of (4.4) we conclude that $p = p_0 - \Pi(R) + \mathcal{R}^{0,2}_{k',m'}$. Then $\widetilde{\psi}(p, p_0) = \underline{\psi}(\Pi(R)) + \mathcal{R}^{1,2}_{k,m}$ with $\underline{\psi}(\varrho) := \widetilde{\psi}(p_0 - \varrho, p_0)$ a C^∞ function with $\underline{\psi}(\varrho) = O(|\varrho|^2)$ for ϱ near 0.

$\square$

Lemma 4.4. *Under the hypotheses and notation of Lemma 4.3, for an* $\boldsymbol{R}'$ *like* $\boldsymbol{R}''$*, for a* $\psi \in C^\infty$ *with* $\psi(\varrho) = O(|\varrho|^2)$ *near 0, we have*

$$K\circ\mathfrak{F} = \psi(\Pi(R)) + \frac{1}{2}\Omega(\mathcal{H}_{p_0}R, R) + \mathcal{R}^{1,2}_{k,m}(\Pi(R), R) + E_P(R) + \boldsymbol{R}', \quad (4.14)$$
$$\boldsymbol{R}' := \sum_{d=2}^{4}\langle B_d(R, \Pi(R)), R^d\rangle + \int_{\mathbb{R}^3} B_5(x, R, R(x), \Pi(R))R^5(x)dx,$$

the B_d *for* $d = 2, ..., 5$ *with similar properties of the functions in Lemma 4.3.*

Proof. We have

$$P(p)R = R + (P(p) - P(p_0))R = R + \mathbf{S}^{1,1}(p - p_0, R) = R + \mathbf{S}^{1,1}(\Pi(R), R).$$

Substituting $P(p)R = R + \mathbf{S}^{1,1}(\Pi(R), R)$ in (4.9) we obtain that $\mathcal{R}^{1,2}_{k,m} + \mathbf{R}''$ is absorbed in $\mathcal{R}^{1,2}_{k,m}(\Pi(R), R) + \mathbf{R}'$. This is elementary to see for the terms with $d \le 4$. We consider the case $d = 5$.

$$\begin{aligned}
&B_5(x, R, R(x), \Pi(R))R^i(x)(\mathbf{S}^{1,1})^{5-i} \\
&= \sum_{j=0}^{5-i} \frac{1}{j!}(\partial_t^j)_{|t=0}[B_5(x, R, tR(x), \Pi(R))]R^i(x)(\mathbf{S}^{1,1})^{5-i} \\
&+ \int_0^1 \frac{(1-t)^{4-i}}{(4-i)!}\partial_t^{5-i}[B_5(x, R, tR(x), \Pi(R))]R^i(x)(\mathbf{S}^{1,1})^{5-i}
\end{aligned}$$

The last term can be absorbed in the $d = 5$ term of $\mathbf{R}'$. Similarly, all the other terms either are absorbed in $\mathbf{R}'$ or, like for instance the $i = j = 0$ term, they are $\mathcal{R}^{1,2}$.
We write $E_P(P(p)R) = E_P(R - P_{N_g(p)}R)$ and use (4.8) for $\mathbf{u} = R$ and $\mathbf{v} = -P_{N_g(p)}R$. We get the sum of $E_P(R)$ with a term which can be absorbed in $\mathcal{R}^{1,2}_{k,m}(\Pi(R), R) + \mathbf{R}'$. We finally focus on

$$\begin{aligned}
&\frac{1}{2}\langle J^{-1}\mathcal{H}_p P(p)R, P(p)R\rangle = \frac{1}{2}\langle \mathcal{D}P(p)R, P(p)R\rangle \\
&- \lambda_j(p)\Pi_j(P(p)R) + \frac{1}{2}\langle \nabla^2 E_P(\Phi_p)P(p)R, P(p)R\rangle.
\end{aligned} \tag{4.15}$$

We have

$$\begin{aligned}
\langle \mathcal{D}P(p)R, P(p)R\rangle &= \langle \mathcal{D}R, R\rangle + \mathcal{R}^{1,2}_{k,m}(\Pi(R), R) \\
\langle \nabla^2 E_P(\Phi_p)P(p)R, P(p)R\rangle &= \langle \nabla^2 E_P(\Phi_{p_0})R, R\rangle + \mathcal{R}^{1,2}_{k,m}(\Pi(R), R) \\
&+ \langle(\nabla^2 E_P(\Phi_p) - \nabla^2 E_P(\Phi_{p_0}))R, R\rangle \\
\lambda_j(p) &= \lambda_j(p_0) + \mathcal{R}^{1,0}(\Pi(R)) + \mathcal{R}^{1,2}_{k,m}(\Pi(R), R) \\
\Pi_j(P(p)R) &= \Pi_j(R) + \mathcal{R}^{1,2}_{k,m}(\Pi(R), R).
\end{aligned}$$

Then we conclude that the right hand side of (4.15) is

$$\begin{aligned}
&\overbrace{\frac{1}{2}\langle(\mathcal{D} - \lambda(p_0)\cdot\Diamond + \nabla^2 E_P(\Phi_{p_0}))R, R\rangle}^{\frac{1}{2}\langle J^{-1}\mathcal{H}_{p_0}R,R\rangle} + \mathcal{R}^{2,0}(\Pi(R)) + \mathcal{R}^{1,2}_{k,m}(\Pi(R), R) \\
&+ \frac{1}{2}\langle(\nabla^2 E_P(\Phi_p) - \nabla^2 E_P(\Phi_{p_0}))R, R\rangle
\end{aligned} \tag{4.16}$$

where the last term can be absorbed in the $d = 2$ term of $\mathbf{R}'$. Setting $\psi(\varrho) = \underline{\psi}(\varrho) + \mathcal{R}^{2,0}(\varrho)$ with the $\mathcal{R}^{2,0}$ in (4.16), we get the desired result. $\square$

We have completed the part of this paper devoted to the Darboux Theorem. The next step consists in the decomposition of R into discrete and continuous modes, and the search of a new coordinate system by an appropriate Birkhoff normal forms argument.

5 Spectral coordinates associated to $\mathcal{H}_{p_0}$

We will consider the operator $\mathcal{H}_{p_0}$, which will be central in our analysis henceforth. We will list now various hypotheses, starting with the spectrum of $\mathcal{H}_{p_0}$ thought as an operator in the natural complexification $L^2(\mathbb{R}^3, \mathbb{C}^{2N})$ of $L^2(\mathbb{R}^3, \mathbb{R}^{2N})$.

(L1) $\sigma_e(\mathcal{H}_{p_0})$ is a union of intervals in $\mathrm{i}\mathbb{R}$ with $0 \notin \sigma_e(\mathcal{H}_{p_0})$ and is symmetric with respect to 0.

(L2) $\sigma_p(\mathcal{H}_{p_0})$ is finite.

(L3) For any eigenvalue $\mathbf{e} \in \sigma_p(\mathcal{H}_{p_0})\backslash\{0\}$ the algebraic and geometric dimensions coincide and are finite.

(L4) There is a number $\mathbf{n} \geq 1$ and positive numbers $0 < \mathbf{e}'_1 \leq \mathbf{e}'_2 \leq ... \leq \mathbf{e}'_{\mathbf{n}}$ such that $\sigma_p(\mathcal{H}_{p_0})$ consists exactly of the numbers $\pm\mathrm{i}\mathbf{e}'_j$ and 0. We assume that there are fixed integers $\mathbf{n}_0 = 0 < \mathbf{n}_1 < ... < \mathbf{n}_{l_0} = \mathbf{n}$ such that $\mathbf{e}'_j = \mathbf{e}'_i$ exactly for i and j both in $(\mathbf{n}_l, \mathbf{n}_{l+1}]$ for some $l \leq l_0$. In this case $\dim\ker(\mathcal{H}_{p_0} - \mathbf{e}'_j) = \mathbf{n}_{l+1} - \mathbf{n}_l$. We assume there exist $N_j \in \mathbb{N}$ such that $N_j + 1 = \inf\{n \in \mathbb{N} : n\mathbf{e}'_j \in \sigma_e(\mathcal{H}_{p_0})\}$. We set $\mathbf{N} = \sup_j N_j$. We assume that $\mathbf{e}'_j \notin \sigma_p(\mathcal{H}_{p_0})$ for all j.

(L5) If $\mathbf{e}'_{j_1} < ... < \mathbf{e}'_{j_i}$ are i distinct λ's, and $\mu \in \mathbb{Z}^k$ satisfies $|\mu| \leq 2N + 3$, then we have
$$\mu_1\mathbf{e}'_{j_1} + \cdots + \mu_k\mathbf{e}'_{j_i} = 0 \iff \mu = 0 \ .$$

The following hypothesis holds quite generally.

(L6) If $\varphi \in \ker(\mathcal{H}_{p_0} - \mathrm{i}\mathbf{e})$ for $\mathrm{i}\mathbf{e} \in \sigma_p(\mathcal{H}_{p_0})$ then $\varphi \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^{2N})$.

By (2.15), $\mathcal{H}_{p_0}\xi = \mathbf{e}\xi$ implies $\mathcal{H}^*_{p_0}J^{-1}\xi = -\mathbf{e}J^{-1}\xi$. Then $\sigma_p(\mathcal{H}_{p_0}) = \sigma_p(\mathcal{H}^*_{p_0})$. We denote it by σ_p.
By general argument we have:

Lemma 5.1. *The following spectral decomposition remains determined:*

$$\begin{aligned} N_g^{\perp}(\mathcal{H}^*_{p_0}) \otimes_{\mathbb{R}} \mathbb{C} &= \left(\oplus_{\mathbf{e}\in\sigma_p\backslash\{0\}} \ker(\mathcal{H}_{p_0} - \mathbf{e})\right) \oplus X_c(p_0) \\ X_c(p_0) &:= \left\{N_g(\mathcal{H}^*_{p_0}) \oplus \left(\oplus_{\mathbf{e}\in\sigma_p\backslash\{0\}} \ker(\mathcal{H}^*_{p_0} - \mathbf{e})\right)\right\}^{\perp} . \end{aligned} \tag{5.1}$$

We denote by P_c the projection on $X_c(p_0)$ associated to (5.1). Set $\mathcal{H} := \mathcal{H}_{p_0}P_c$.
The following hypothesis is important to solve the homological equations in the Birkhoff normal forms argument.

(L7) We have $R_{\mathcal{H}} \circ \Diamond^i_j \in C^{\omega}(\rho(\mathcal{H}), B(\Sigma_n, \Sigma_n))$ for any $n \in \mathbb{N}$, any $j = 1, ..., n_0$ and for any $i = 0, 1$, where $\rho(\mathcal{H}) = \mathbb{C}\backslash\sigma_e(\mathcal{H}_{p_0})$.

For the examples in Sect. 7, (L7) can be checked with standard arguments.
We discuss now the choice of a good frame of eigenfunctions.

Lemma 5.2. *It is possible to choose eigenfunctions $\xi' \in \ker(\mathcal{H}_{p_0} - \mathrm{i}\mathbf{e}_j')$ so that $\Omega(\xi_j', \overline{\xi}_k') = 0$ for $j \neq k$ and $\Omega(\xi_j', \overline{\xi}_j') = -\mathrm{i}s_j$ with $s_j \in \{1, -1\}$. We have $\Omega(\xi_j', \xi_k') = 0$ for all j and k. We have $\Omega(\xi, f) = 0$ for any eigenfunction ξ and any $f \in X_c(p_0)$.*

Proof. First of all, if $\lambda, \mu \in \sigma_p(\mathcal{H}_{p_0})$ are two eigenvalues with $\lambda \neq 0$ and given two associated eigenfunctions ξ_μ and ξ_λ

$$\begin{aligned} \langle J^{-1}\xi_\lambda, \overline{\xi}_\mu \rangle &= \frac{1}{\lambda} \langle J^{-1}\mathcal{H}_{p_0}\xi_\lambda, \overline{\xi}_\mu \rangle = -\frac{1}{\lambda} \langle \mathcal{H}_{p_0}^* J^{-1}\xi_\lambda, \overline{\xi}_\mu \rangle \\ &= -\frac{1}{\lambda} \langle J^{-1}\xi_\lambda, \mathcal{H}_{p_0}\overline{\xi}_\mu \rangle = -\frac{\overline{\mu}}{\lambda} \langle J^{-1}\xi_\lambda, \overline{\xi}_\mu \rangle, \end{aligned} \tag{5.2}$$

where for the second equality we used (2.15) and for the last one the fact that $\mathcal{H}_{p_0}\xi = \mu\xi$ implies $\mathcal{H}_{p_0}\overline{\xi} = \overline{\mu}\overline{\xi}$. Then, for $\mathbf{e}_j \neq \mathbf{e}_k$ and associated eigenfunctions ξ_j and ξ_k we get $\Omega(\xi_j, \overline{\xi}_k) = 0$. Notice that by a similar argument we have $\Omega(\xi_\lambda, \xi_\mu) = -\frac{\mu}{\lambda}\Omega(\xi_\lambda, \xi_\mu)$ and so $\Omega(\xi_j', \xi_k') \equiv 0$.
Since $\mathcal{H}_{p_0}\xi = \mathbf{e}\xi$ implies $\mathcal{H}_{p_0}^* J^{-1}\xi = -\mathbf{e}J^{-1}\xi$, for any eigenfunction ξ of $\mathcal{H}_{p_0}$ then $J^{-1}\xi$ is an eigenfunction of $\mathcal{H}_{p_0}^*$. By the definition of $X_c(p_0)$ in (5.1), we conclude $\Omega(\xi, f) = \langle J^{-1}\xi, f \rangle = 0$ for any $f \in X_c(p_0)$.
Let $\mathrm{i}\mathbf{e} \in \mathrm{i}\mathbb{R}\backslash\{0\}$ be an eigenvalue. By the above discussion, the Hermitian form $\langle \mathrm{i}J^{-1}\xi, \overline{\eta} \rangle$ is non degenerate in $\ker(\mathcal{H}_{p_0} - \mathrm{i}\mathbf{e})$. Then we can find a basis such that $\langle \mathrm{i}J^{-1}\eta_j, \overline{\eta}_k \rangle = -|a_j|\mathrm{sign}(a_j)\delta_{jk}$, for appropriate non zero numbers $a_j \in \mathbb{R}$. Then set $\xi' = \sqrt{|a_j|}\eta_j$. ☐

We set $\xi_j = \xi_j'$ and $\mathbf{e}_j = \mathbf{e}_j'$ if $s_j = 1$.
We set $\overline{\xi}_j = \xi_j'$ and $\mathbf{e}_j = -\mathbf{e}_j'$ if $s_j = -1$.
Notice that if $f \in X_c(p_0)$ then also $\overline{f} \in X_c(p_0)$. This implies that for $R \in N_g^\perp(\mathcal{H}_{p_0}^*) \otimes_{\mathbb{R}} \mathbb{C}$ with real entries, that is if $R = \overline{R}$, then we have

$$R(x) = \sum_{j=1}^{\mathbf{n}} z_j \xi_j(x) + \sum_{j=1}^{\mathbf{n}} \overline{z}_j \overline{\xi}_j(x) + f(x), \quad f \in X_c(p_0). \tag{5.3}$$

with $f = \overline{f}$.
By Lemma 5.2 we have, for the s_j of Lemma 5.2,

$$\frac{1}{2}\Omega(\mathcal{H}_{p_0}R, R) = \sum_{j=1}^{\mathbf{n}} \mathbf{e}_j |z_j|^2 + \frac{1}{2}\Omega(\mathcal{H}_{p_0}f, f) =: H_2. \tag{5.4}$$

Consider the map $R \to (z, f)$ obtained from (5.3). In terms of the pair (z, f), the Fréchet derivative R' can be expressed as

$$R' = \sum_{j=1}^{\mathbf{n}} (dz_j \xi_j + d\overline{z}_j \overline{\xi}_j) + f'.$$

We have

$$\Omega(R', R') = -\mathrm{i}\sum_{j=1}^{\mathbf{n}} dz_j \wedge d\overline{z}_j + \Omega(f', f'). \tag{5.5}$$

For a function F independent of τ and Π let us decompose X_F as of spectral decomposition (5.3):

$$X_F = \sum_{j=1}^{\mathbf{n}} (X_F)_{z_j} \xi_j(x) + \sum_{j=1}^{\mathbf{n}} (X_F)_{\overline{z}_j} \overline{\xi}_j(x) + (X_F)_f, \quad (X_F)_f \in X_c(p_0).$$

By $i_{X_F}\Omega = dF$ and by

$$\begin{aligned} dF &= \partial_{z_j} F dz_j + \partial_{\overline{z}_j} F d\overline{z}_j + \langle \nabla_f F, f' \rangle \\ i_{X_F}\Omega &= -\mathrm{i}(X_F)_{z_j} d\overline{z}_j + \mathrm{i}(X_F)_{\overline{z}_j} dz_j + \langle J^{-1}(X_F)_f, f' \rangle, \end{aligned}$$

we get

$$(X_F)_{z_j} = \mathrm{i}\partial_{\overline{z}_j} F \ , \quad (X_F)_{\overline{z}_j} = -\mathrm{i}\partial_{z_j} F \ , \quad (X_F)_f = J\nabla_f F.$$

This implies

$$\{F, G\} := dF(X_G) = \mathrm{i}\partial_{z_j} F \partial_{\overline{z}_j} G - \mathrm{i}\partial_{\overline{z}_j} F \partial_{z_j} G + \langle \nabla_f F, J\nabla_f G \rangle. \tag{5.6}$$

Hence, for H_2 defined in (5.4), for $z = (z_1, ..., z_{\mathbf{n}})$, using standard multi index notation and by (2.15), we have:

$$\{H_2, z^\mu \overline{z}^\nu\} = -\mathrm{i}\mathbf{e} \cdot (\mu - \nu) z^\mu \overline{z}^\nu \ ; \quad \{H_2, \langle J^{-1}\varphi, f \rangle\} = \langle J^{-1}\mathcal{H}\varphi, f \rangle. \tag{5.7}$$

5.1 Flows in spectral coordinates

We restate Lemma 3.8 for a special class of transformations.

Lemma 5.3. *Consider*

$$\chi = \sum_{|\mu+\nu|=M_0+1} b_{\mu\nu}(\Pi(f)) z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle J^{-1} B_{\mu\nu}(\Pi(f)), f \rangle \tag{5.8}$$

with $b_{\mu\nu}(\varrho) = \mathcal{R}^{i,0}_{r,M}(\varrho)$ and $B_{\mu\nu}(\varrho) = \boldsymbol{S}^{i,0}_{r,M}(\varrho)$ with $i \in \{0,1\}$ fixed and $r, M \in \mathbb{N}$ sufficiently large and with

$$\overline{b}_{\mu\nu} = b_{\nu\mu} \ , \quad \overline{B}_{\mu\nu} = B_{\nu\mu}, \tag{5.9}$$

(so that χ is real valued for $f = \overline{f}$). Then we have what follows.

(1) Consider the vectorfield X_χ defined with respect to Ω_0. Then, summing on repeated indexes (with the equalities defining the field X^{st}_χ), we have:

$$\begin{aligned} &(X_\chi)_{z_j} = \mathrm{i}\partial_{\overline{z}_j}\chi =: (X^{st}_\chi)_{z_j} \ , \quad (X_\chi)_{\overline{z}_j} = -\mathrm{i}\partial_{z_j}\chi =: (X^{st}_\chi)_{\overline{z}_j} \ , \\ &(X_\chi)_f = \partial_{\Pi_j(f)}\chi \, P^*_c(p_0) J \Diamond_j f + (X^{st}_\chi)_f \ \textit{where} \ (X^{st}_\chi)_f := z^\mu \overline{z}^\nu B_{\mu\nu}(\Pi(f)). \end{aligned}$$

(2) *Denote by* ϕ^t *the flow of* X_χ *provided by Lemma 3.8 and set* $(z^t, f^t) = (z, f) \circ \phi^t$. *Then we have*

$$z^t = z + \mathcal{Z}(t) \qquad f^t = e^{Jq(t)\cdot\Diamond}(f + \boldsymbol{S}(t)) \tag{5.10}$$

where, for (k, m) *with* $k \in \mathbb{Z} \cap [0, r - (m+1)\boldsymbol{d}]$ *and* $1 \le m \le M$, *for* $B_{\Sigma_{-k}}$ *a sufficiently small neighborhood of 0 in* $\Sigma_{-k} \cap X_c(p_0)$ *and for* $B_{\mathbb{C}^{\mathbf{n}}}$ *(resp.*$B_{\mathbb{R}^{n_0}}$*) a neighborhood of 0 in* $\mathbb{C}^{\mathbf{n}}$ *(resp.*$\mathbb{R}^{n_0}$*)*

$$\begin{aligned} &\boldsymbol{S} \in C^m((-2,2) \times B_{\mathbb{C}^{\mathbf{n}}} \times B_{\Sigma_{-k}} \times B_{\mathbb{R}^{n_0}}, \Sigma_k) \\ &q \in C^m((-2,2) \times B_{\mathbb{C}^{\mathbf{n}}} \times B_{\Sigma_{-k}} \times B_{\mathbb{R}^{n_0}}, \mathbb{R}^{n_0}) \\ &\mathcal{Z} \in C^m((-2,2) \times B_{\mathbb{C}^{\mathbf{n}}} \times B_{\Sigma_{-k}} \times B_{\mathbb{R}^{n_0}}, \mathbb{C}^{\mathbf{n}}), \end{aligned} \tag{5.11}$$

with for fixed C

$$\begin{aligned} &|q(t, z, f, \varrho)| \le C(|z| + \|f\|_{\Sigma_{-k}})^{M_0+1} \\ &|\mathcal{Z}(t, z, f, \varrho)| + \|\boldsymbol{S}(t, z, f, \varrho)\|_{\Sigma_k} \le C(|z| + \|f\|_{\Sigma_{-k}})^{M_0}. \end{aligned} \tag{5.12}$$

We have $\boldsymbol{S}(t, z, f, \varrho) = \boldsymbol{S}_1(t, z, f, \varrho) + \boldsymbol{S}_2(t, z, f, \varrho)$ *with*

$$\begin{aligned} &\boldsymbol{S}_1(t, z, f, \varrho) = \int_0^t (X_\chi^{st})_f \circ \phi^{t'} dt' \\ &\|\boldsymbol{S}_2(t, z, f, \varrho)\|_{\Sigma_k} \le C(|z| + \|f\|_{\Sigma_{-k}})^{2M_0+1}(|z| + \|f\|_{\Sigma_{-k}} + |\varrho|)^i. \end{aligned} \tag{5.13}$$

(3) *The flow* ϕ^t *is canonical: for* s, s', k *as in Lemma 3.8, the map* $\phi^t \in C^l(\mathcal{U}^{s'}_{\varepsilon_1,k}, \widetilde{\mathcal{P}}^s)$ *satisfies* $\phi^{t*}\Omega_0 = \Omega_0$ *in* $C^\infty(\mathcal{U}^{s'}_{\varepsilon_2,k}, B^2(\widetilde{\mathcal{P}}^{s'}, \mathbb{R}))$ *for* $\varepsilon_2 > 0$ *sufficiently small.*

Proof. First of all notice that χ does not depend on τ and Π so that the only nonzero component of X_χ is $(X_\chi)_R = J\nabla_R\chi$. The latter is of the form indicated in claim (1) by a direct computation. Claim (2) follows now by Lemma 3.8.
To prove Claim (3) we need to make rigorous the following formal computation

$$\frac{d}{dt}\phi^{t*}\Omega_0 = \phi^{t*}L_{X_\chi}\Omega_0 = \phi^{t*}di_{X_\chi}\Omega_0 = \phi^{t*}d^2\chi = 0.$$

To make sense of this we can proceed as in Corollary 3.12. We skip the proof. $\square$

Lemma 5.4. *Consider a transformation* $\mathfrak{F} = \mathfrak{F}_1 \circ \cdots \circ \mathfrak{F}_L$ *like in Lemma 4.1 and with* $m_1 = 2$ *and for fixed* r *and* M *sufficiently large. Denote by* (k', m') *the pair* (k, m) *of Lemma 4.4 and consider a pair* (k, m) *with* $k \le k'$ *and* $m \le m' - (2\boldsymbol{N} + 5)$. *Set* $H' := K \circ \mathfrak{F}$. *Consider decomposition* (5.3). *Then on a domain* $\mathcal{U}^s_{\varepsilon,k}$ *like* (3.20) *we have*

$$H' = \psi(\Pi(f)) + H_2' + \boldsymbol{R}\ , \tag{5.14}$$

for a $\psi \in C^\infty$ *with* $\psi(\varrho) = O(|\varrho|^2)$ *near 0 and with what follows.*

(1) We have

$$H_2' = \sum_{\substack{|\mu+\nu|=2 \\ \mathbf{e}\cdot(\mu-\nu)=0}} a_{\mu\nu}(\Pi(f)) z^\mu \overline{z}^\nu + \frac{1}{2}\langle J^{-1}\mathcal{H}_{p_0} f, f\rangle. \tag{5.15}$$

(2) We have $\boldsymbol{R} = \boldsymbol{R}_{-1} + \boldsymbol{R}_0 + \boldsymbol{R}_1 + \boldsymbol{R}_2 + \mathcal{R}^{1,2}_{k,m+2}(\Pi(f), f) + \boldsymbol{R}_3 + \boldsymbol{R}_4$, *with:*

$$\boldsymbol{R}_{-1} = \sum_{\substack{|\mu+\nu|=2 \\ \mathbf{e}\cdot(\mu-\nu)\neq 0}} a_{\mu\nu}(\Pi(f)) z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=1} z^\mu \overline{z}^\nu \langle J^{-1} G_{\mu\nu}(\Pi(f)), f\rangle;$$

For $\mathbf{N}$ *as in (L4) of this section,*

$$\boldsymbol{R}_0 = \sum_{|\mu+\nu|=3}^{2\mathbf{N}+1} z^\mu \overline{z}^\nu a_{\mu\nu}(\Pi(f));$$

$$\boldsymbol{R}_1 = \sum_{|\mu+\nu|=2}^{2\mathbf{N}} z^\mu \overline{z}^\nu \langle J^{-1} G_{\mu\nu}(\Pi(f)), f\rangle;$$

$$\boldsymbol{R}_2 = \langle \mathbf{B}_2(\Pi(f)), f^2\rangle \text{ with } \mathbf{B}_2(0) = 0$$

where $f^d(x)$ *represents schematically* $d-$*products of components of* f*;*

$$\boldsymbol{R}_3 = \sum_{\substack{|\mu+\nu|= \\ =2N+2}} z^\mu \overline{z}^\nu a_{\mu\nu}(z, f, \Pi(f)) + \sum_{\substack{|\mu+\nu|= \\ =2N+1}} z^\mu \overline{z}^\nu \langle J^{-1} G_{\mu\nu}(z, f, \Pi(f)), f\rangle;$$

$$\boldsymbol{R}_4 = \sum_{d=2}^{4} \langle B_d(z, f, \Pi(f)), f^d\rangle + \int_{\mathbb{R}^3} B_5(x, z, f, f(x), \Pi(f)) f^5(x) dx$$
$$+ \widehat{\boldsymbol{R}}_2(z, f, \Pi(f)) + E_P(f) \text{ with } B_2(0, 0, \varrho) = 0.$$

(3) For $\delta_j := (\delta_{1j}, ..., \delta_{mj})$,

$$\begin{aligned} &a_{\mu\nu}(0) = 0 \text{ for } |\mu+\nu| = 2 \text{ with } (\mu,\nu) \neq (\delta_j, \delta_j) \text{ for all } j, \\ &a_{\delta_j\delta_j}(0) = \lambda_j(\omega_0), \\ &G_{\mu\nu}(0) = 0 \text{ for } |\mu+\nu| = 1\ . \end{aligned} \tag{5.16}$$

These $a_{\mu\nu}(\varrho)$ *and* $G_{\mu\nu}(x,\varrho)$ *are* C^m *in all variables with* $G_{\mu\nu}(\cdot,\varrho) \in C^m(\mathrm{U}, \Sigma_k(\mathbb{R}^3, \mathbb{C}^{2N}))$*, for a small neighborhood* U *of* $(0,0,0)$ *in* $\mathbb{C}^{\mathbf{n}} \times (\Sigma_{-k} \cap X_c(p_0)) \times \mathbb{R}^{n_0}$ *(the space of the* (z, f, ϱ)*), and they satisfy symmetries analogous to* (5.9).

(4) We have $a_{\mu\nu}(z,\varrho) \in C^m(\mathrm{U}, \mathbb{C})$.

(5) $G_{\mu\nu}(\cdot, z, \varrho) \in C^m(\mathrm{U}, \Sigma_k(\mathbb{R}^3, \mathbb{C}^{2N})))$.

(6) $B_d(\cdot, z, f, \varrho) \in C^m(\mathrm{U}, \Sigma_k(\mathbb{R}^3, B((\mathbb{C}^{2N})^{\otimes d}, \mathbb{R})))$, *for* $2 \le d \le 4$. $\mathbf{B}_2(\cdot, \varrho)$ *satisfies the same property.*

(7) Let $\zeta \in \mathbb{C}^{2N}$. *Then for* $B_5(\cdot, z, f, \zeta, \varrho)$ *we have (the derivatives are not in the holomorphic sense)*

$$\text{for } |l| \le m \ , \ \|\nabla^l_{z,f,\zeta,\varrho} B_5(z, f, \zeta, \varrho)\|_{\Sigma_k(\mathbb{R}^3, B((\mathbb{R}^{2N})^{\otimes 5}, \mathbb{R})} \le C_l.$$

(8)

$$\begin{aligned} &\widehat{\boldsymbol{R}}_2 \in C^m(\mathrm{U}, \mathbb{C}), \\ &|\widehat{\boldsymbol{R}}_2(z, f, \varrho)| \le C(|z| + \|f\|_{\Sigma_{-k}})\|f\|^2_{\Sigma_{-k}}; \end{aligned} \tag{5.17}$$

Proof. We need to express R in terms of (z, f) using (5.3) inside (4.14). We have $\Pi(R) = \Pi(f) + \mathcal{R}^{0,2}(R)$. Then, succinctly,

$$\mathcal{R}^{1,2}_{k',m'}(\Pi(R), R) = \sum_{a+b=2}^{2\mathbf{N}+1} \frac{1}{a!b!} \langle \nabla^a_\varrho \nabla^b_R \mathcal{R}^{1,2}_{k',m'}(\Pi(f), 0), (\mathcal{R}^{0,2}(R))^a R^{b\otimes}\rangle +$$
$$\sum_{\substack{a+b\\=2\mathbf{N}+2}} \int_0^1 \frac{(1-t)^{2\mathbf{N}+1}}{a!b!} \langle \nabla^a_\varrho \nabla^b_R \mathcal{R}^{1,2}_{k',m'}(\Pi(f) + t\mathcal{R}^{0,2}(R), tR), (\mathcal{R}^{0,2}(R))^a R^{b\otimes}\rangle dt,$$

with (k', m') the pair (k, m) of Lemma 4.4. We substitute (5.3), that is $R = z \cdot \xi + \overline{z} \cdot \overline{\xi} + f$. For $m \le m' - (2\mathbf{N}+2)$ and $k \le k'$, the terms from the $R^{b\otimes}$ of degree in f at most 1, go into $\mathbf{R}_i$ with $i = -1, 0, 1, 3$ and H_2'. For $m \le m' - (2\mathbf{N}+4)$, the remaining terms are absorbed in $\mathcal{R}^{1,2}_{k',m+2}(\Pi(f), f) + \widehat{\mathbf{R}}_2(z, f, \Pi(f))$.
We focus now on the $d = 5$ term in (4.14). We substitute $R = z \cdot \xi + \overline{z} \cdot \overline{\xi} + f$. This schematically yields, for a $\widetilde{B}_5$ satisfying claim (7) with the pair (m', k'),

$$\sum_{j=0}^{5} \int_{\mathbb{R}^3} \widetilde{B}_5(x, z, f, f(x), \Pi(f))(z \cdot \xi + \overline{z} \cdot \overline{\xi})^{5-j} f^j(x) dx. \tag{5.18}$$

For $j = 5$ we get a term that can be absorbed in the B_5 term in $\mathbf{R}_4$. Expand the $j < 5$ terms in (5.18) as

$$\sum_{i=0}^{4-j} \int_{\mathbb{R}^3} \frac{1}{i!} (\partial_t^i)_{|t=0} \widetilde{B}_5(x, z, f, tf(x), \Pi(f))(z \cdot \xi + \overline{z} \cdot \overline{\xi})^{5-j} f^{i+j}(x) dx +$$
$$\int_{\mathbb{R}^3} \frac{1}{(4-j)!} \int_0^1 \partial_t^{5-j} [\widetilde{B}_5(x, z, f, tf(x), \Pi(f))](z \cdot \xi + \overline{z} \cdot \overline{\xi})^{5-j} f^5(x) dx.$$

go into the B_d term in $\mathbf{R}_4$ The last term fits in the B_5 term in $\mathbf{R}_4$ by $m \le m' - 5$. The terms in the first line go into the B_d of $\mathbf{R}_4$ for $d = i + j \ge 2$. The terms

with $i+j<2$ can be treated like the $\mathcal{R}^{1,2}_{k',m'}(\Pi(R),R)$ for $m\le m'-(2\mathbf{N}+5)$ and $k\le k'$.

We focus on $E_P(R)=E_P(z\cdot\xi+\overline{z}\cdot\overline{\xi}+f)$. We use Lemma 4.2 for $\mathbf{v}=f$ and $\mathbf{u}=z\cdot\xi+\overline{z}\cdot\overline{\xi}$. Then

$$\begin{aligned}
&E_P(R)=E_P(f)+E_P(z\cdot\xi+\overline{z}\cdot\overline{\xi})+\\
&\int_{\mathbb{R}^3}dx\sum_{j=0}^{3}\int_{[0,1]^2}\frac{t^j}{j!}(\partial_t^{j+1})_{|t=0}\partial_s[B(|s(z\cdot\xi+\overline{z}\cdot\overline{\xi})+tf|_1^2)]dtds\\
&+\int_{\mathbb{R}^3}dx\int_{[0,1]^2}dtds\int_0^t\partial_\tau^5\partial_s[B(|s(z\cdot\xi+\overline{z}\cdot\overline{\xi})+\tau f|_1^2)]\frac{(t-\tau)^3}{3!}d\tau.
\end{aligned}$$

By $B(0)=B'(0)=0$, we have $E_P(z\cdot\xi+\overline{z}\cdot\overline{\xi})=\mathcal{R}^{0,4}(R)$. It is easy to conclude that this term easily fits into $\mathbf{R}_0+\mathbf{R}_3$. Similarly, the $j=0$ term fits in $\mathbf{R}_1+\mathbf{R}_3$. The $j\ge1$ terms fit in the B_{j+1} term in $\mathbf{R}_4$. The last line fits in the B_5 term in $\mathbf{R}_4$.

The symmetries (5.9) for the coefficients in $H_2'+\mathbf{R}_{-1}+\mathbf{R}_0+\mathbf{R}_1$ are an elementary consequence of the fact that H' is real valued.

$\square$

Remark 5.5. Given a Hamiltonian H' expanded as in Lemma 5.4 and given a transformation $\mathfrak{F}$, we cannot obtain the expansion of Lemma 5.4 for $H'\circ\mathfrak{F}$ analysing one by one the terms of the expansion of H'. This works in the set up of [6, 7] but not here (see in particular the discussion on the exponential under formula (6.31) later).

6 Birkhoff normal forms

In this section we arrive at the main result of the paper.

6.1 Homological equations

We consider $a^{(\ell)}_{\mu\nu}(\varrho)\in C^{\widehat{m}}(U,C)$ for $k_0\in\mathbb{N}$ a fixed number and U a neighborhood of 0 in $\mathbb{R}^{n_0}$. Then we set

$$H_2^{(\ell)}(\varrho):=\sum_{\substack{|\mu+\nu|=2\\ \mathbf{e}\cdot(\mu-\nu)=0}}a^{(\ell)}_{\mu\nu}(\varrho)z^\mu\overline{z}^\nu+\frac{1}{2}\langle J^{-1}\mathcal{H}f,f\rangle. \tag{6.1}$$

$$\mathbf{e}_j(\varrho):=a^{(\ell)}_{\delta_j\delta_j}(\varrho),\quad \mathbf{e}(\varrho)=(\lambda_1(\varrho),\cdots,\lambda_m(\varrho)). \tag{6.2}$$

We assume $\mathbf{e}_j(0)=\mathbf{e}_j$ and $a^{(\ell)}_{\mu\nu}(0)=0$ if $(\mu,\nu)\ne(\delta_j,\delta_j)$ for all j, with δ_j defined in (5.16).

Definition 6.1. A function $Z(z, f, \varrho)$ is in normal form if $Z = Z_0 + Z_1$ where Z_0 and Z_1 are finite sums of the following type:

$$Z_1 = \sum_{\mathbf{e}(0)\cdot(\nu-\mu)\in\sigma_e(\mathcal{H}_{p_0})} z^\mu \overline{z}^\nu \langle J^{-1} G_{\mu\nu}(\varrho), f\rangle \tag{6.3}$$

with $G_{\mu\nu}(x, \varrho) \in C^m(U, \Sigma_k(\mathbb{R}^3, \mathbb{C}^{2N}))$ for fixed $k, m \in \mathbb{N}$ and $U \subseteq \mathbb{R}^{n_0}$ a neighborhood of 0;

$$Z_0 = \sum_{\mathbf{e}(0)\cdot(\mu-\nu)=0} g_{\mu\nu}(\varrho) z^\mu \overline{z}^\nu \tag{6.4}$$

and $g_{\mu\nu}(\varrho) \in C^m(U, \mathbb{C})$. We assume furthermore that the above coefficients satisfy the symmetries in (5.9): that is $\overline{g}_{\mu\nu} = g_{\nu\mu}$ and $\overline{G}_{\mu\nu} = G_{\nu\mu}$.

Lemma 6.2. *We consider $\chi = \chi(b, B)$ with*

$$\chi(b, B) = \sum_{|\mu+\nu|=M_0+1} b_{\mu\nu} z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle J^{-1} B_{\mu\nu}, f\rangle \tag{6.5}$$

for $b_{\mu\nu} \in \mathbb{C}$ and $B_{\mu\nu} \in \Sigma_{\widehat{k}}(\mathbb{R}^3, \mathbb{C}^{2N}) \cap X_c(p_0)$ with $\widehat{k} \in \mathbb{N}$, satisfying the symmetries in (5.9). *Here we interpret the polynomial χ as a function with parameters $b = (b_{\mu\nu})$ and $B = (B_{\mu\nu})$. Denote by $X_{\widehat{k}}$ the space of the pairs (b, B). Let us also consider given polynomials with $K = K(\varrho)$ and $\widetilde{K} = \widetilde{K}(\varrho, b, B)$ where:*

$$K(\varrho) := \sum_{|\mu+\nu|=M_0+1} k_{\mu\nu}(\varrho) z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle J^{-1} K_{\mu\nu}(\varrho), f\rangle, \tag{6.6}$$

with $k_{\mu\nu}(\varrho) \in C^{\widehat{m}}(U, \mathbb{C})$ and $K_{\mu\nu}(\varrho) \in C^{\widehat{m}}(U, \Sigma_{\widehat{k}}(\mathbb{R}^3, \mathbb{C}^{2N}) \cap X_c(p_0))$ for U a neighborhood of 0 in $\mathbb{R}^{n_0}$, satisfying the symmetries in (5.9)*;*

$$\begin{aligned} \widetilde{K}(\varrho, b, B) := &\sum_{|\mu+\nu|=M_0+1} \widetilde{k}_{\mu\nu}(\varrho, b, B) z^\mu \overline{z}^\nu \\ &+ \sum_{i=0}^{1} \sum_{j=1}^{n_0} \sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle J^{-1} \Diamond_j^i K^i_{j\mu\nu}(\varrho, b, B), f\rangle, \end{aligned} \tag{6.7}$$

with $\widetilde{k}_{\mu\nu} \in C^{\widehat{m}}(U \times X_{\widehat{k}}, \mathbb{R})$ and $\widetilde{K}^i_{j\mu\nu} \in C^{\widehat{m}}(U \times X_{\widehat{k}}, \Sigma_{\widehat{k}}(\mathbb{R}^3, \mathbb{C}^{2N}) \cap X_c(p_0))$, satisfying the symmetries in (5.9). *Suppose also that the sums* (6.6) *and* (6.7) *do not contain terms in normal form and that $\widetilde{K}(0, b, B) = 0$. Then there exists a neighborhood $V \subseteq U$ of 0 in $\mathbb{R}^{n_0}$ and a unique choice of functions $(b(\varrho), B(\varrho)) \in C^{\widehat{m}}(V, X_{\widehat{k}})$ such that for $\chi(\varrho) = \chi(b(\varrho), B(\varrho))$, $\widetilde{K}(\varrho) = \widetilde{K}(\varrho, b(\varrho), B(\varrho))$ we have*

$$\left\{\chi(\varrho), H_2^{(\ell)}(\varrho)\right\}^{st} = K(\varrho) + \widetilde{K}(\varrho) + Z(\varrho) \tag{6.8}$$

where $\{\cdots\}^{st}$ is the bracket (5.6) *for ϱ fixed and where $Z(\varrho)$ is in normal form and homogeneous of degree $M_0 + 1$ in (z, f).*

Proof. Summing on repeated indexes, by (5.7) we get

$$\begin{aligned} \{H_2^{(\ell)}, \chi\}^{st} &= -\mathrm{i}\mathbf{e}(\varrho)\cdot(\mu-\nu) z^\mu \overline{z}^\nu b_{\mu\nu}(\varrho) \\ &- z^\mu\overline{z}^\nu \langle f, J^{-1}(\mathrm{i}\mathbf{e}(\varrho)\cdot(\mu-\nu) - \mathcal{H})B_{\mu\nu}(\varrho)\rangle + \widehat{K}(\varrho, b(\varrho), B(\varrho)), \end{aligned} \tag{6.9}$$

$$\begin{aligned} \widehat{K}(\varrho, b, B) :&= \sum_{\substack{|\mu+\nu|=2 \\ (\mu,\nu)\neq(\delta_j,\delta_j)\ \forall\ j}} a_{\mu\nu}^{(\ell)}(\varrho)\Big[\sum_{|\mu'+\nu'|=M_0+1} \{z^\mu\overline{z}^\nu, z^{\mu'}\overline{z}^{\nu'}\} b_{\mu'\nu'} \\ &+ \sum_{|\mu'+\nu'|=M_0} \{z^\mu\overline{z}^\nu, z^{\mu'}\overline{z}^{\nu'}\}\langle J^{-1}B_{\mu'\nu'}, f\rangle\Big]. \end{aligned} \tag{6.10}$$

$\widehat{K}$ is a homogeneous polynomial of the same type of the above ones and we have $\widehat{K}(0, b, B) = 0$. In particular, $\widehat{K}$ satisfies the symmetries (5.9) by (for $f = \overline{f}$)

$$\begin{aligned} &(a_{\mu\nu}^{(\ell)} b_{\mu'\nu'}\{z^\mu\overline{z}^\nu, z^{\mu'}\overline{z}^{\nu'}\})^* = a_{\nu\mu}^{(\ell)} b_{\nu'\mu'}\{z^\nu\overline{z}^\mu, z^{\nu'}\overline{z}^{\mu'}\} \\ &(a_{\mu\nu}^{(\ell)}\langle J^{-1}B_{\mu'\nu'}, f\rangle\{z^\mu\overline{z}^\nu, z^{\mu'}\overline{z}^{\nu'}\})^* = a_{\nu\mu}^{(\ell)}\langle J^{-1}B_{\nu'\mu'}, f\rangle\{z^\nu\overline{z}^\mu, z^{\nu'}\overline{z}^{\mu'}\} \end{aligned}$$

which follow by $(\mathrm{i}\partial_{z_j}F\partial_{\overline{z}_j}G - \mathrm{i}\partial_{\overline{z}_j}F\partial_{z_j}G)^* = \mathrm{i}\partial_{z_j}F^*\partial_{\overline{z}_j}G^* - \mathrm{i}\partial_{\overline{z}_j}F^*\partial_{z_j}G^*$, where in these formulas $a^* = \overline{a}$, and by the symmetries (5.9) for χ and for $H_2^{(\ell)}$. Denote by $\widehat{Z}(\varrho, b, B)$ the sum of monomials in normal form of $\widetilde{K}$ and set $\mathbf{K} := \widetilde{K} + \widehat{K} - \widehat{Z}$. We look at

$$\begin{aligned} &- \mathrm{i}\mathbf{e}(\varrho)\cdot(\mu-\nu) z^\mu\overline{z}^\nu b_{\mu\nu} - z^\mu\overline{z}^\nu\langle f, J^{-1}(\mathrm{i}\mathbf{e}(\varrho)\cdot(\mu-\nu) - \mathcal{H})B_{\mu\nu}\rangle \\ &+ \mathbf{K}\ (\varrho, b, B) + K(\varrho) = 0 \end{aligned} \tag{6.11}$$

that is at

$$\begin{aligned} &k_{\mu\nu}(\varrho) + \mathbf{k}_{\mu\nu}(\varrho, b, B) - b_{\mu\nu}(\varrho)\mathrm{i}\mathbf{e}(\varrho)\cdot(\mu-\nu) = 0 \\ &B_{\mu\nu}(\varrho) = -R_{\mathcal{H}}(\mathrm{i}\mathbf{e}(\varrho)\cdot(\mu-\nu))\left[K_{\mu\nu}(\varrho) + \mathbf{K}_{\mu\nu}(\varrho, b, B)\right], \end{aligned} \tag{6.12}$$

with $\mathbf{k}_{\mu\nu}$ and $\mathbf{K}_{\mu\nu}$ the coefficients of $\mathbf{K}$. Notice that when $\mathbf{k}_{\mu\nu}(0, b, B) = 0$ and $\mathbf{K}_{\mu\nu}(0, b, B) = 0$, for $\varrho = 0$ there is a unique solution $(b, B) \in X_{\widehat{k}}$ given by

$$b_{\mu\nu}(0) = \frac{k_{\mu\nu}(0)}{\mathrm{i}\mathbf{e}\cdot(\mu-\nu)}, \quad B_{\mu\nu}(0) = -R_{\mathcal{H}}(\mathrm{i}\mathbf{e}\cdot(\mu-\nu))K_{\mu\nu}(0). \tag{6.13}$$

Lemma 6.2 is then a consequence of the Implicit Function Theorem by Hypothesis (L7) in Sect. 5. □

In the particular case $M_0 = 1$ we need a slight variation of Lemma 6.2.

Lemma 6.3. *Suppose now $M_0 = 1$ and assume the notation of Lemma 6.2, assuming $K(0) = 0$, $\widetilde{K}(0, 0, 0) = 0$ and $\nabla_{b,B}\widetilde{K}(0, 0, 0) = 0$. We furthermore consider function $a_{\mu\nu}^{\mu'\nu'} \in C^{\widehat{m}}(U \times X_{\widehat{k}}, \mathbb{C})$ with $|a_{\mu\nu}^{\mu'\nu'}(\varrho, b, B)| \leq C\|(b, B)\|_{X_{\widehat{k}}}$ and we set*

$$
\begin{aligned}
\Big\{\chi(\varrho), H_2^{(\ell)}(\varrho)\Big\}^{\widetilde{st}} &= \Big\{\chi(\varrho), H_2^{(\ell)}(\varrho)\Big\}^{st} \\
&+ \sum_{\substack{|\mu+\nu|=1 \\ |\mu'+\nu'|=1}} a_{\mu\nu}^{\mu'\nu'}(\varrho, b(\varrho), B(\varrho)) z^\mu \overline{z}^\nu \langle \mathcal{H} B_{\mu'\nu'}(\varrho), f\rangle.
\end{aligned} \tag{6.14}
$$

Then, the same conclusions of Lemma 6.2 hold for

$$
\Big\{\chi(\varrho), H_2^{(\ell)}(\varrho)\Big\}^{\widetilde{st}} = K(\varrho) + \widetilde{K}(\varrho) + Z(\varrho). \tag{6.15}
$$

Proof. Like above we get to

$$
\begin{aligned}
& k_{\mu\nu}(\varrho) + \mathbf{k}_{\mu\nu}(\varrho, b, B) - b_{\mu\nu} \mathbf{ie}(\varrho) \cdot (\mu - \nu) = 0 \\
& B_{\mu\nu} = -R_{\mathcal{H}}(\mathbf{ie}(\varrho) \cdot (\mu - \nu))[K_{\mu\nu}(\varrho) + \mathbf{K}_{\mu\nu}(\varrho, b, B) + \sum_{\mu'\nu'} a_{\mu\nu}^{\mu'\nu'}(\varrho, b, B) \mathcal{H} B_{\mu'\nu'}].
\end{aligned}
$$

For $(\varrho, b, B) = (0, 0, 0)$ both sides are 0. Then Lemma 6.3 follows by Implicit Function Theorem. ∎

6.2 The Birkhoff normal forms

Our goal in this section is to prove the following result where N is as of (L4) in Sect. 5.

Theorem 6.4. *For any integer $2 \le \ell \le 2\mathbf{N} + 1$ we have transformations $\mathfrak{F}^{(\ell)} = \mathfrak{F}_1 \circ \phi_2 \circ \ldots \circ \phi_\ell$, with $\mathfrak{F}_1$ the transformation in Corollary 3.12 the ϕ_j's like in Lemma 5.3, such that the conclusions of Lemma 5.4 hold, that is such that we have the following expansion*

$$
H^{(\ell)} := K \circ \mathfrak{F}^{(\ell)} = \psi(\Pi(f)) + H_2^{(\ell)} + \mathcal{R}_{k,m+2}^{1,2}(\Pi(f), f) + \sum_{j=-1}^{4} \boldsymbol{R}_j^{(\ell)},
$$

with $H_2^{(\ell)}$ of the form (5.15) and with the following additional properties:

(i) $\boldsymbol{R}_{-1}^{(\ell)} = 0$*;*

(ii) all the nonzero terms in $\boldsymbol{R}_0^{(\ell)}$ with $|\mu + \nu| \le \ell$ are in normal form, that is $\lambda \cdot (\mu - \nu) = 0$;

(iii) all the nonzero terms in $\boldsymbol{R}_1^{(\ell)}$ with $|\mu + \nu| \le \ell - 1$ are in normal form, that is $\lambda \cdot (\mu - \nu) \in \sigma_e(\mathcal{H}_{p_0})$.

Proof. The proof of Theorem 6.4 is by induction. There are two distinct parts in the proof, [7, 6, 2]. Here we follow the ordering of [2]. In the first part we assume that for some $\ell \ge 2$ the statement of the theorem is true, and we show that it continues to be true for $\ell + 1$. The proof of case $\ell = 2$, which presents some additional complications, is dealt in the second part.

In the proof we will get polynomials (5.8) with $M_0 = 1, ..., 2\mathbf{N}$ with decreasing (r, M) as M_0 increases. Nonetheless, in view of the fact that in Lemma 3.7 the n is arbitrarily large and that (r, M) decreases by a fixed amount at each step, these (r, M) are arbitrarily large. This is exploited in Theorem 6.5 later.

The step $\ell + 1 > 2$. We can assume that $H^{(\ell)}$ have the desired properties for indexes (k', m') (instead of (k, m)) arbitrarily large. We consider the representation (5.14) for $H^{(\ell)}$ and we set $\mathbf{h} = H^{(\ell)}(z, f, \varrho)$ replacing $\Pi(f)$ with ϱ in (5.14). Then $\mathbf{h} = H^{(\ell)}(z, f, \varrho)$ is $C^{2\mathbf{N}+2}$ near 0 in $\mathcal{P}^{s_0} = \{(\varrho, R)\}$ for $m' \geq 2\mathbf{N} + 2$ for $s_0 > \max\{\mathrm{ord}(\mathcal{H}_{p_0}), 3/2\}$ by Lemma 5.4. So we have equalities

$$a_{\mu\nu}(\varrho) = \frac{1}{\mu!\nu!}\partial_z^\mu \partial_{\overline{z}}^\nu \mathbf{h}_{|(z,f,\varrho)=(0,0,\varrho)} \ , \quad |\mu + \nu| \leq 2\mathbf{N} + 1, \tag{6.16}$$

$$J^{-1}G_{\mu\nu}(\varrho) = \frac{1}{\mu!\nu!}\partial_z^\mu \partial_{\overline{z}}^\nu \nabla_f \mathbf{h}_{|(z,f,\varrho)=(0,0,\varrho)} \ , \quad |\mu + \nu| \leq 2\mathbf{N}. \tag{6.17}$$

We consider now a yet unknown χ as in (5.8) with $M_0 = \ell$, $i = 0$, $M = m'$ and $r = k'$. Set $\phi := \phi^1$, where ϕ^t is the flow of Lemma 5.3. We are seeking χ such that $H^{(\ell)} \circ \phi$ satisfies the conclusions of Theorem 6.4 for $\ell + 1$.
We know that $H^{(\ell)} \circ \phi$ satisfies the conclusions of Lemma 5.4. Therefore, to prove the induction step, all we need to do is to check that the expansion of $H^{(\ell)} \circ \phi$ satisfies $\mathbf{R}_{-1} = 0$ and that the only terms in $\mathbf{R}_0$ and $\mathbf{R}_1$ of degree $\leq \ell + 1$ are in normal form. We have

$$H_2^{(\ell)} \circ \phi = H_2^{(\ell)} + \int_0^1 \{H_2^{(\ell)}, \chi\}^{st} \circ \phi^t dt + \int_0^1 (\partial_{\varrho_j} a_{\mu\nu} z^\mu \overline{z}^\nu \{\Pi_j(f), \chi\}) \circ \phi^t dt. \tag{6.18}$$

By (6.9)–(6.10) we have for $\varrho = \Pi(f)$

$$\begin{aligned} \{H_2^{(\ell)}, \chi\}^{st} = &-\mathrm{i} \sum_{|\mu+\nu|=\ell+1} \mathbf{e}^{(\ell)}(\varrho) \cdot (\mu - \nu) z^\mu \overline{z}^\nu b_{\mu\nu}(\varrho) \\ &- \sum_{|\mu+\nu|=\ell} z^\mu \overline{z}^\nu \langle J^{-1}(\mathrm{i}\mathbf{e}^{(\ell)}(\varrho) \cdot (\mu - \nu) - \mathcal{H}) B_{\mu\nu}(\varrho), f\rangle + \\ &\sum_{\substack{|\mu+\nu|=2 \\ (\mu,\nu)\neq(\delta_j,\delta_j) \ \forall \ j}} a_{\mu\nu}^{(\ell)}(\varrho) \big[\sum_{|\mu'+\nu'|=\ell+1} \{z^\mu \overline{z}^\nu, z^{\mu'} \overline{z}^{\nu'}\} b_{\mu'\nu'}(\varrho) \\ &+ \sum_{|\mu'+\nu'|=\ell} \{z^\mu \overline{z}^\nu, z^{\mu'} \overline{z}^{\nu'}\} \langle J^{-1} B_{\mu'\nu'}(\varrho), f\rangle \big]. \end{aligned} \tag{6.19}$$

By Lemma 5.3 for $M_0 = \ell$, $i = 0$, $M = m'$ and $r = k'$ for first and last formula and by the proof of Lemma 3.8, in particular by (3.35), we have

$$\begin{aligned} &z \circ \phi^t = z + \mathcal{R}^{0,\ell}_{k'',m'}(t, \Pi(f), R) \, , \, \Pi(f) \circ \phi^t = \Pi(f) + \mathcal{R}^{0,\ell+1}_{k'',m'}(t, \Pi(f), R) \, , \\ &f \circ \phi^t = e^{J\mathcal{R}^{0,\ell+1}_{k'',m'}(t,\Pi(f),R)\cdot\Diamond}(f + \mathbf{S}^{0,\ell}_{k'',m'}(t, \Pi(f), R)) \end{aligned} \tag{6.20}$$

for $k'' \le k' - (m'+1)\mathbf{d}$. Then, substituting (6.20) in (6.19) we get, if $k \le k'' - \text{ord}(\mathcal{H}_{p_0})$, where $\text{ord}(\mathcal{H}_{p_0}) \le \max\{\text{ord}(\mathcal{D}), \mathbf{d}\}$, for $1 \le m \le m'$ and exploiting that an $\mathcal{R}^{0,2\ell}_{k,m}$ is also an $\mathcal{R}^{0,\ell+2}_{k,m}$ for $\ell \ge 2$,

$$\int_0^1 \{H_2^{(\ell)}, \chi\}^{st} \circ \phi^t dt = \{H_2^{(\ell)}, \chi\}^{st} + \mathcal{R}^{0,\ell+2}_{k,m}(\Pi(f), R). \tag{6.21}$$

We have

$$\{\Pi_j(f), \chi\} = \sum_{k=1}^{n_0} \{\Pi_j(f), \Pi_k(f)\} \partial_{\Pi_k(f)} \chi + \sum_{|\mu'+\nu'|=\ell} z^{\mu'} \overline{z}^{\nu'} \langle P_c^*(p_0) \Diamond_j f, B_{\mu'\nu'} \rangle.$$

We have, for $P_d(p_0) = 1 - P_c(p_0)$ the projection on the direct sum of $N_g(\mathcal{H}_{p_0})$ and the complement of $X_c(p_0)$ in (5.1), and using $JP_c^*(p_0) = P_c(p_0)J$ which follows from (2.15),

$$\begin{aligned} &\{\Pi_i(f), \Pi_j(f)\} = \langle P_c^*(p_0) \Diamond_i f, JP_c^*(p_0) \Diamond_j f \rangle \\ &= \langle \Diamond_i f, P_d(p_0) J \Diamond_j f \rangle = \mathcal{R}^{0,2}(f). \end{aligned} \tag{6.22}$$

Notice also that, for $B_{\mu\nu} \in \Sigma_{k'}$ independent of $\Pi(f)$ and for $|\mu + \nu| = \ell$, we have

$$\begin{aligned} &\{\Pi_i(f), z^\mu \overline{z}^\nu \langle J^{-1} B_{\mu\nu}, f \rangle\} = z^\mu \overline{z}^\nu \langle P_c^*(p_0) \Diamond_i f, B_{\mu\nu} \rangle = \\ &z^\mu \overline{z}^\nu \langle f, \Diamond_i B_{\mu\nu} \rangle - z^\mu \overline{z}^\nu \langle P_d^*(p_0) \Diamond_i f, B_{\mu\nu} \rangle = \mathcal{R}^{0,\ell+1}_{k'-\mathbf{d},\infty}(R) + \mathcal{R}^{0,\ell+1}(R). \end{aligned} \tag{6.23}$$

By (6.22)–(6.23) we conclude that $\{\Pi_j(f), \chi\} = \mathcal{R}^{0,\ell+1}_{k'-\mathbf{d},m'}(\Pi(f), R)$. By (6.20) we get for $m \le m'$

$$\begin{aligned} &\{\Pi_j(f), \chi\} \circ \phi^t = \mathcal{R}^{0,\ell+1}_{k'-\mathbf{d},m'} \left(\Pi(f) + \mathcal{R}^{0,\ell+1}_{k'',m'}(t, \Pi(f), R), S \right), \\ &\text{for } S := e^{J\mathcal{R}^{0,\ell+1}_{k'',m'}(t,\Pi(f),R)\cdot\Diamond} \left(R + \mathbf{S}^{0,\ell}_{k'',m'}(t, \Pi(f), R) \right). \end{aligned}$$

Then

$$\{\Pi_j(f), \chi\} \circ \phi^t = \mathcal{R}^{0,\ell+1}_{k''-m'\mathbf{d},m'}(t, \Pi(f), R). \tag{6.24}$$

By (6.20) and (6.24) the last term in (6.18) is $\mathcal{R}^{0,\ell+2}_{k,m}(\Pi(f), R)$ for $k \le k'' - m'\mathbf{d}$. This and (6.21) yield for $k = \min\{k' - (2m'+1)\mathbf{d}, k' - (m'+1)\mathbf{d} - \text{ord}(\mathcal{H}_{p_0})\}$

$$H_2^{(\ell)} \circ \phi = H_2^{(\ell)} + \{H_2^{(\ell)}, \chi\}^{st} + \mathcal{R}^{0,\ell+2}_{\tilde{k},m}(\Pi(f), R). \tag{6.25}$$

A second observation is that $\mathbf{h} = (H^{(\ell)} \circ \phi)(z, f, \varrho)$ is $C^{2\mathbf{N}+2}$ in $\mathcal{P}^{s_0} = \{(\varrho, R)\}$ for $m \ge 2\mathbf{N}+2$. We can compute again the corresponding coefficients in (6.16)–(6.17). Because of (5.12), for $|\mu + \nu| \le \ell$ in (6.16) and for $|\mu + \nu| \le \ell - 1$ in (6.17) these coefficients are the same of $\mathbf{h} = H^{(\ell)}(z, f, \varrho)$.
A third observation is that for $j = 3, 4$ we have for $\mathbf{k} = \mathbf{R}_j^{(\ell)} \circ \phi$

$$\begin{aligned} &\partial_z^\mu \partial_{\overline{z}}^\nu \mathbf{k}_{|(0,0,\varrho)} = 0 \text{ for } |\mu| + |\nu| \le \ell + 1 \\ &\partial_z^\mu \partial_{\overline{z}}^\nu \nabla_f \mathbf{k}_{|(0,0,\varrho)} = 0 \text{ for } |\mu| + |\nu| \le \ell. \end{aligned} \tag{6.26}$$

By Lemma 3.10 for $l=m$, $s=k$ and $r=k'$, we have for $k\le k'-(2m+1)\mathbf{d}$

$$\Pi_j(f)\circ\phi=\Pi_j(f)\circ\phi_0+\mathcal{R}^{0,2\ell+1}_{k,m}(\Pi(f),R), \tag{6.27}$$

with $\phi_0=\phi_0^1$ and ϕ_0^t the flow defined as in Lemma 3.10 using the field X^{st}_χ. Then we have

$$\Pi_j(f)\circ\phi_0=\Pi_j(f)+\int_0^1\left\langle\Diamond_j(X^{st}_\chi)_f(\Pi(f),R\circ\phi_0^t),f\circ\phi_0^t\right\rangle dt. \tag{6.28}$$

By the definition of X^{st}_χ and by formulas (6.20) for ϕ_0^t, which are simpler because there are no phase factors, by $|\mu+\nu|=\ell$ the integrand in (6.28) is

$$\begin{aligned}&\left(z+\mathcal{R}^{0,\ell}_{k'',m}(t,\Pi(f),R)\right)^\mu\left(\overline{z}+\mathcal{R}^{0,\ell}_{k'',m}(t,\Pi(f),R)\right)^\nu\\&\times\left\langle\Diamond_jB_{\mu,\nu}(\Pi(f)),f+\mathbf{S}^{0,\ell}_{k'',m}(t,\Pi(f),R)\right\rangle\\&=z^\mu\overline{z}^\nu\langle\Diamond_jB_{\mu,\nu}(\Pi(f)),f\rangle+\mathcal{R}^{0,2\ell}_{k'',m}(t,\Pi(f),R).\end{aligned}$$

Then for $k\le k''$ we have

$$\Pi_j(f)\circ\phi_0=\Pi_j(f)+\langle\Diamond_j(X^{st}_\chi)_f,f\rangle+\mathcal{R}^{0,2\ell}_{k,m}(\Pi(f),R). \tag{6.29}$$

By $\ell\ge2$ we have $2\ell\ge\ell+2$ and so $\mathcal{R}^{0,2\ell}_{k,m}$ is an $\mathcal{R}^{0,\ell+2}_{k,m}$.
By $\psi(\varrho)=O(|\varrho|^2)$ near 0, we conclude that

$$\psi(\Pi(f))\circ\phi=\psi(\Pi(f))+\widetilde{K}'+\mathcal{R}^{1,\ell+2}_{k,m}(\Pi(f),R), \tag{6.30}$$

with $\widetilde{K}'$ a polynomial as in (6.7) with $M_0=\ell$, with $\widetilde{K}'(0,b,B)=0$ and $(\widehat{k},\widehat{m})=(k',m')$ satisfying. Notice that it was to get the last equality, which follows from (6.29), that we introduced the flow ϕ_0^t.
We now focus on $\mathbf{R}_2$. We have by (6.20)

$$\begin{aligned}&\mathbf{R}_2\circ\phi=\langle\mathbf{B}_2(\Pi(f')),(f')^2\rangle= \\&\langle\mathbf{B}_2\left(\Pi(f)+\mathcal{R}^{0,\ell+1}_{k,m}(\Pi(f),R)\right),\left(e^{J\mathcal{R}^{0,\ell+1}_{k'',m'}(\Pi(f),R)\cdot\Diamond}(f+\mathbf{S}^{0,\ell}_{k'',m'}(\Pi(f),R))\right)^2\rangle.\end{aligned} \tag{6.31}$$

In our present set up the exponential $e^{J\mathcal{R}^{0,\ell+1}_{k'',m'}\cdot\Diamond}$ cannot be moved to the $\mathbf{B}_2$ by a change of variables in the integral as in [6]. Fortunately we know already that $H^{(\ell)}\circ\phi$ has the expansion of Lemma 5.4 and that all we need to do is to compute some derivatives of $\mathbf{R}_2\circ\phi$.
Using the expansion in (6.31) and formula (5.13), for $i=0$ now, we set

$$\begin{aligned}&\mathfrak{R}_2:=\langle\mathbf{B}_2(\Pi(f)),(f+\mathbf{S}^{i,\ell}_{k'',m'}(\Pi(f),R))^2\rangle=\\&\left\langle\mathbf{B}_2(\Pi(f)),\left[f+\int_0^1(X^{st}_\chi)_f\circ\phi^tdt+\mathbf{S}^{i,2\ell+1}_{k'',m'}(\Pi(f),R)\right]^2\right\rangle=\\&\langle\mathbf{B}_2(\Pi(f)),f^2\rangle+2\int_0^1\langle\mathbf{B}_2(\Pi(f)),(X^{st}_\chi)_f\circ\phi^t\ f\rangle dt+\mathcal{R}^{i,2\ell}_{k'',m'}(\Pi(f),R).\end{aligned} \tag{6.32}$$

We have that $\mathbf{k} = \mathbf{R}_2 \circ \phi - \mathfrak{R}_2$ is $C^{\ell+1}$ and satisfies (6.26). Hence the analysis of $\mathbf{R}_2 \circ \phi$ reduces to that of $\mathfrak{R}_2$. By (6.20), for $k \leq k''$, $m \leq m' - 1$ and $\ell > 1$ we have

$$\int_0^1 X_\chi^{st} \circ \phi^t dt = X_\chi^{st} + \mathbf{S}^{0,2\ell-1}_{k'',m'-1}(\Pi(f), R) = X_\chi^{st} + \mathbf{S}^{0,\ell+1}_{k,m}(\Pi(f), R). \tag{6.33}$$

This implies

$$\begin{aligned} &\mathfrak{R}_2 = \langle \mathbf{B}_2(\Pi(f)), f^2 \rangle + \widetilde{K}'' + \mathcal{R}^{0,\ell+2}_{k,m}(\Pi(f), R) \quad , \\ &\widetilde{K}'' := 2\langle \mathbf{B}_2(\Pi(f)), f(X_\chi^{st})_f \rangle. \end{aligned} \tag{6.34}$$

Then $\widetilde{K}''$ is a polynomial like in (6.7) for the pair $(\widehat{k}, \widehat{m}) = (k', m')$ satisfying $\widetilde{K}''(0, b, B) = 0$ by $B_2(\varrho) = 0$ for $\varrho = 0$.
By (6.20) and for the pullback of the term $\mathcal{R}^{1,2}_{k',m'+2}(\Pi(f), f)$ in Lemma 5.4 we have for $\varrho = \Pi(f)$

$$\begin{aligned} &\mathcal{R}^{1,2}_{k',m'+2}(\Pi(f'), f') = \mathcal{R}^{1,2}_{k',m'+2}(\varrho, f') \\ &+ \int_0^1 (\nabla_\varrho \mathcal{R}^{1,2}_{k',m'+2})(\varrho + t\mathcal{R}^{0,\ell+1}_{k'',m'+2}(\varrho, f), f') \cdot \mathcal{R}^{0,\ell+1}_{k'',m'}(\varrho, f) dt \\ &= \mathcal{R}^{1,2}_{k',m'+2}(\varrho, f') + \mathcal{R}^{0,\ell+3}_{k,m}(\varrho, R) \end{aligned} \tag{6.35}$$

for $k \leq k'' - m\mathbf{d}$ and $m \leq m'$, by elementary analysis of the second line.
Applying again (6.20) we have

$$\begin{aligned} \mathcal{R}^{1,2}_{k',m'+2}(\varrho, f') &= \mathcal{R}^{1,2}_{k',m'+2}\left(\varrho, e^{J\mathcal{R}^{0,\ell+1}_{k'',m'}(\varrho,R)\cdot\Diamond}\left(f + \mathbf{S}^{0,\ell}_{k'',m'}(\varrho, R)\right)\right) \\ &= \mathcal{R}^{1,2}_{k',m'+2}\left(\varrho, f + \mathbf{S}^{0,\ell}_{k'',m'}(\varrho, R)\right) + \mathcal{R}^{1,\ell+2}_{k,m}(\varrho, R) \end{aligned} \tag{6.36}$$

for $k \leq k'' - m\mathbf{d}$ and $m \leq m' - 1$. Next, by Lemma 5.3, (5.13) and by (6.33), we have $\mathbf{S}^{0,\ell}_{k'',m'}(\varrho, R) = (X_\chi^{st})_f + \mathbf{S}^{0,\ell+1}_{k'',m'-1}(\varrho, R)$ and

$$\begin{aligned} &\mathcal{R}^{1,2}_{k',m'+2}\left(\varrho, f + (X_\chi^{st})_f + \mathbf{S}^{0,\ell+1}_{k'',m}(\varrho, R)\right) = \mathcal{R}^{1,2}_{k',m'+2}(\varrho, f) + \\ &\int_0^1 \langle \nabla_R \mathcal{R}^{1,2}_{k',m'+2}\left(\varrho, f + t(X_\chi^{st})_f + t\mathbf{S}^{0,\ell+1}_{k'',m}(\varrho, R)\right), (X_\chi^{st})_f + \mathbf{S}^{0,\ell+1}_{k'',m}(\varrho, R) \rangle dt \\ &= \mathcal{R}^{1,2}_{k',m'+2}(\varrho, f) + \langle \nabla_f \mathcal{R}^{1,2}_{k',m'+2}(\varrho, f), (X_\chi^{st})_f \rangle + \mathcal{R}^{1,\ell+2}_{k,m}(\varrho, R) \end{aligned}$$

where we have used $\ell \geq 2$, $k \leq k'' \leq k'$ and $m \leq m' - 1$. Notice that we have that $\mathcal{R}^{1,2}_{k',m'+2}(\varrho, f)$ is an $\mathcal{R}^{1,2}_{k,m+2}(\varrho, f)$. Finally we have

$$\begin{aligned} &\langle \nabla_f \mathcal{R}^{1,2}_{k',m'+2}(\varrho, f), (X_\chi^{st})_f \rangle = \widetilde{K}''' + \overline{\mathbf{R}}_2 \quad , \\ &\widetilde{K}''' := \langle \nabla_f^2 \mathcal{R}^{1,2}_{k',m'+2}(\varrho, 0) f, (X_\chi^{st})_f \rangle, \end{aligned} \tag{6.37}$$

with $\overline{\mathbf{R}}_2$ a term we can absorb in $\widehat{\mathbf{R}}_2$ and with $\widetilde{K}'''$ like in (6.7) for the pair $(\widehat{k}, \widehat{m}) = (k', m')$ satisfying $\widetilde{K}'''(0, b, B) = 0$.

We set

$$\mathbf{R}_0^{(\ell)} + \mathbf{R}_1^{(\ell)} = Z' + K + \mathbf{R}_{01} \ , \tag{6.38}$$

where: Z' is the sum of the monomials in normal form of degree $\leq \ell + 1$; K, which is like in (6.6), is the sum of the the monomials of degree equal to $\ell + 1$ not in normal form; $\mathbf{R}_{01}$ is the sum of the monomials of degree $> \ell + 1$. By induction there are no monomials not in normal form of degree $\leq \ell$ so that each of the monomials of the lhs of (6.38) go into exactly one of the three terms of the rhs.
We define Z'' and $\widetilde{K}$ by setting

$$\widetilde{K}' + \widetilde{K}'' + \widetilde{K}''' = Z'' + \widetilde{K}, \tag{6.39}$$

collecting in Z'' all monomials of the lhs in normal form (all of degree $\ell + 1$) and in $\widetilde{K}$ all monomials of the lhs not in normal form. Here $\widetilde{K}$ is like in (6.7) for $(\widehat{k}, \widehat{m}) = (k', m')$ with $\widetilde{K}(0, b, B) = 0$.
Applying Lemma 6.2 for $(\widehat{k}, \widehat{m}) = (k', m')$ we can choose χ such that for $Z = Z' + Z''$ we have

$$\{H_2^{(\ell)}, \chi\}^{st} + Z + K + \widetilde{K} = 0. \tag{6.40}$$

Then $H^{(\ell+1)} := H^{(\ell)} \circ \phi$ satisfies the conclusions of Theorem 6.4 for $\ell + 1$.

The step $\ell + 1 = 2$. Set $H^{(1)} = K \circ \mathfrak{F}_1$. We are seeking a transformation ϕ as in the previous part such that $H^{(2)} := H^{(1)} \circ \phi$ has term $\mathbf{R}_{-1}^{(2)} = 0$ in its expansion in Lemma 5.4. The argument is similar to the previous one, but this time χ has degree $\ell+1$ with $\ell = 1$. So the steps in the previous argument where we exploited $\ell \geq 2$ need to be reframed. We know that $H^{(1)}$ satisfies Lemma 5.4 for $L = 1$ for some pair that we denote by (k', m') rather than (k, m).
The proof of (6.21) is different from the previous one. By (3.40) we have for some (k, m) appropriately smaller than (k', m')

$$\{H_2^{(1)}, \chi\}^{st} \circ \phi^t = \{H_2^{(1)}, \chi\}^{st} \circ \phi_0^t + \mathcal{R}^{0,4}_{k,m}(\Pi(f), R). \tag{6.41}$$

The following linear transformation

$$(Z, \overline{Z}, F) \to \begin{pmatrix} \mathrm{i}\nu_j b_{\mu\nu}(\Pi(f)) \frac{Z^\mu \overline{Z}^\nu}{\overline{Z}_j} + \mathrm{i}\nu_j \frac{Z^\mu \overline{Z}^\nu}{\overline{Z}_j} \langle J^{-1} B_{\mu\nu}(\Pi(f)), F\rangle \\ -\mathrm{i}\mu_j b_{\mu\nu}(\Pi(f)) \frac{Z^\mu \overline{Z}^\nu}{Z_j} - \mathrm{i}\mu_j \frac{Z^\mu \overline{Z}^\nu}{Z_j} \langle J^{-1} B_{\mu\nu}(\Pi(f)), F\rangle \\ B_{\mu\nu}(\Pi(f)) Z^\mu \overline{Z}^\nu \end{pmatrix}$$

depends linearly on $(b(\varrho), B(\rho))$, for $\varrho = \Pi(f)$. Then

$$z_j \circ \phi_0^t = z_j + a_j(t, b, B) \cdot z + b_j(t, b, B) \cdot \overline{z} + \sum_{\mu\nu} c_{j\mu\nu}(t, b, B) \langle J^{-1} B_{\mu\nu}, f\rangle \tag{6.42}$$

for $a_j, b_j \in C^\infty([0,1]\times X_{k'}, \mathbb{C}^{\mathbf{n}})$ with $|a_j|+|b_j| \le C\|(b,B)\|_{X_{k'}}$ and $c_{j\mu\nu} \in C^\infty([0,1]\times X_{k'}, \mathbb{C})$. Similarly

$$f\circ\phi_0^t = f + \mathbf{a}(t,b,B)\cdot z + \ \mathbf{b}(t,b,B)\cdot\overline{z} + \sum_{\mu\nu} \mathbf{c}_{\mu\nu}(t,b,B)\langle J^{-1}B_{\mu\nu}, f\rangle \tag{6.43}$$

with $\mathbf{a},\mathbf{b} \in C^\infty([0,1]\times X_{k'}, \Sigma_{k'}^{\mathbf{n}})$ with $\|\mathbf{a}\|_{\Sigma_{k'}^{\mathbf{n}}} + \|\mathbf{b}\|_{\Sigma_{k'}^{\mathbf{n}}} \le C\|(b,B)\|_{X_{k'}}$ and $\mathbf{c}_{\mu\nu} \in C^\infty([0,1]\times X_{k'}, \Sigma_{k'})$. These coefficients satisfy appropriate symmetries that ensure $\overline{f\circ\phi_0^t} = f\circ\phi_0^t$.
We have

$$\{H_2^{(1)},\chi\}^{st}\circ\phi_0^t = \{H_2^{(1)},\chi\}^{st}(\Pi(f), R\circ\phi_0^t) + \mathcal{R}^{1,4}_{k,m}(t,\Pi(f),R). \tag{6.44}$$

To compute $\{H_2^{(1)},\chi\}^{st}(\Pi(f), R\circ\phi_0^t)$ we replace the R in (6.19) with $R\circ\phi_0^t$. The coordinates of the latter can be expressed in terms of R by (6.42)–(6.43). When we substitute (z,f) in (6.19) using (6.42)–(6.43), by an elementary computation we obtain

$$\begin{aligned}&\{H_2^{(1)},\chi\}^{st}(\varrho, R\circ\phi_0^t) = \{H_2^{(1)},\chi\}^{st}(\varrho, R)\\ &+ \sum_{\substack{|\mu+\nu|=1\\ |\mu'+\nu'|=1}} a_{\mu\nu}^{\mu'\nu'}(t,\varrho,b(\varrho),B(\varrho)) z^\mu \overline{z}^\nu \langle \mathcal{H}B_{\mu\nu}(\varrho), f\rangle + A^t + \underline{\mathbf{R}}^t.\end{aligned}$$

Here:

- $a_{\mu\nu}^{\mu'\nu'}(t,\varrho,b,B) \in C^{m'}$ with $a_{\mu\nu}^{\mu'\nu'}(t,0,0,0)=0$;
- we have

$$\begin{aligned}A^t = &\sum_{|\mu+\nu|=2} \alpha_{\mu\nu}(t,\varrho,b(\varrho),B(\varrho)) z^\mu\overline{z}^\nu\\ &+ \sum_{l=0}^{1}\sum_{j=1}^{n_0}\sum_{|\mu+\nu|=1} z^\mu\overline{z}^\nu \langle \Diamond_j^l A^l_{\mu\nu}(t,\varrho,b(\varrho),B(\varrho)), f\rangle,\end{aligned}$$

  $\alpha_{\mu\nu}(t,\varrho,b,B)$ and $A^l_{\mu\nu}(t,\varrho,b,B)$ are $C^{m'}$ with for $i=2$

$$|\alpha_{\mu\nu}(t,\varrho,b,B)| + \|A^l_{\mu\nu}(t,\varrho,b,B)\|_{\Sigma_{k'}} \le C\|(b,B)\|^i_{X_{k'}}; \tag{6.45}$$

- $\underline{\mathbf{R}}^t(\varrho,z,f)$ is C^m in $(t,\varrho,z,f) \in \mathbb{R}^{n_0+1}\times\mathbb{C}^{\mathbf{n}}\times\Sigma_{-k}$ with (ϱ,z,f) near $(0,0,0)$, with for $i=2$

$$|\underline{\mathbf{R}}^t| \le C\|(b,B)\|^2_{X_{k'}}\|f\|^2_{\Sigma_{-k}}. \tag{6.46}$$

Then, in the notation of Lemma 6.3

$$\int_0^1 \{H_2^{(1)},\chi\}^{st}\circ\phi_0^t dt = \{H_2^{(1)},\chi\}^{\widetilde{st}} + A + \underline{\mathbf{R}} + \mathcal{R}^{1,4}_{k,m}(\Pi(R),R), \tag{6.47}$$

with $A=\int_0^1 A^t dt$ and $\underline{\mathbf{R}}=\int_0^1 \underline{\mathbf{R}}^t dt$ are like A^1 and $\underline{\mathbf{R}}^1$. Then, using also (6.41), we get the following analogue of (6.25):

$$H_2^{(1)}\circ\phi = H_2^{(1)} + \{H_2^{(1)},\chi\}^{\widetilde{st}} + A + \underline{\mathbf{R}} + \mathcal{R}^{0,4}_{k,m}(\Pi(f),R). \tag{6.48}$$

(6.27) remains true also for $\ell=1$. We consider (6.28) and expand

$$\langle \Diamond_j (X_\chi^{st})_f(\Pi(f), R\circ\phi_0^t), f\circ\phi_0^t\rangle = \langle \Diamond_j (X_\chi^{st})_f(\Pi(f),R),f\rangle + A^t + \mathbf{R}^t,$$

with A^t and $\mathbf{R}^t$ like the previous ones but such that (6.45)–(6.46) hold for $i=1$. This yields

$$\Pi_j(f)\circ\phi_0 = \Pi_j(f) + A' + \underline{\mathbf{R}}'. \tag{6.49}$$

Here $\underline{\mathbf{R}}'$ is like $\underline{\mathbf{R}}^1$ such that (6.46) holds for $i=1$. A' is like A^1 such that (6.45) holds for $i=1$.
By $\psi(\varrho)=O(|\varrho|^2)$ near 0 and (6.27) we get the first equality in

$$\begin{aligned} &\psi(\Pi(f))\circ\phi = \psi(\Pi(f))\circ\phi_0 + \mathcal{R}^{1,3}_{k,m}(\Pi(f),R) \\ &= \psi(\Pi(f)) + \widetilde{K}' + \mathcal{R}^{1,2}_{k',m'}(\Pi(f),f) + \mathcal{R}^{1,3}_{k,m}(\Pi(f),R), \end{aligned} \tag{6.50}$$

where $\widetilde{K}' = \mathcal{R}^{1,2}_{k',m'}(\Pi(f),R)$ is a polynomial in R as in (6.7) with $\widetilde{K}'(0,b,B)=0$. The second line in (6.50) follows by $\psi(\varrho)=O(|\varrho|^2)$, by the fact that $\psi(\varrho)$ is smooth and by (6.49). Notice that by choosing $m\le m'-2$ we have $\mathcal{R}^{1,2}_{k',m'}(\Pi(f),f)=\mathcal{R}^{1,2}_{k,m+2}(\Pi(f),f)$.
The discussion of $\mathbf{R}\circ\phi$ is similar to the previous one after (6.31) . This time, though, by (3.40) we write

$$\int_0^1 X_\chi^{st}\circ\phi^t dt = \int_0^1 X_\chi^{st}\circ\phi_0^t dt + \mathbf{S}^{0,3}_{k,m}(\Pi(f),R). \tag{6.51}$$

By (6.42)–(6.43) we get

$$\int_0^1 X_\chi^{st}\circ\phi_0^t dt = X_\chi^{st} + \mathbf{A} \text{ in } \mathcal{P}^{k'}, \tag{6.52}$$

with $(z,f)\to\mathbf{A}(\varrho,z,f)$ linear, with $C^{m'}$ dependence in ϱ and with

$$\|\mathbf{A}(\varrho,z,f)\|_{\mathcal{P}^{k'}} \le C\|(b(\varrho),B(\varrho))\|_{X_{k'}}(|z|+\|f\|_{\Sigma_{-k'}}). \tag{6.53}$$

This yields, for $\mathfrak{R}_2$ defined as in (6.32),

$$\begin{aligned} \mathfrak{R}_2 = \left\langle \mathbf{B}_2(\Pi(f)), \left[f+\int_0^1 (X_\chi^{st})_f\circ\phi_0^t dt\right]^2\right\rangle + \mathcal{R}^{1,3}_{k,m}(\Pi(f),R) = \\ \langle \mathbf{B}_2(\Pi(f)), f^2\rangle + 2\langle \mathbf{B}_2(\Pi(f)), f\mathbf{A}\rangle + \langle \mathbf{B}_2(\Pi(f)), \mathbf{A}^2\rangle + \mathcal{R}^{1,3}_{k,m}(\Pi(f),R), \end{aligned}$$

where we have used $\mathbf{B}_2(0)=0$ for the reminder.
We have

$$2\langle \mathbf{B}_2(\Pi(f)), f\mathbf{A}\rangle + \langle \mathbf{B}_2(\Pi(f)),\mathbf{A}^2\rangle = \widetilde{K}'' + \underline{\mathbf{R}}'',$$

with $\underline{\mathbf{R}}''$ like $\underline{\mathbf{R}}$ and with $\widetilde{K}''$ like (6.7) with $\widetilde{K}''(0,b,B)=0$, by $\mathbf{B}_2(0)=0$, and with $(\widehat{k},\widehat{m})=(k',m')$. Summing up, we have

$$\mathfrak{R}_2=\langle \mathbf{B}_2(\Pi(f)),f^2\rangle+\widetilde{K}''+\underline{\mathbf{R}}''+\mathcal{R}^{1,3}_{k,m}(\Pi(f),R). \tag{6.54}$$

Notice that the reduction of $\mathbf{R}_2\circ\phi$ to $\mathfrak{R}_2$ continues to hold also for $\ell=1$.
We consider $\mathcal{R}^{1,2}_{k',m'+2}\circ\phi$ from the $\mathcal{R}^{1,2}_{k',m'+2}$ term in the expansion of $\mathbf{R}$ in Lemma 5.4. Then, by (6.35) and by (6.51)–(6.52), for $\varrho=\Pi(f)$ we have

$$\mathcal{R}^{1,2}_{k',m'+2}(\Pi(f'),f')=\mathcal{R}^{1,2}_{k',m'+2}(\varrho,f+(X^{st}_\chi)_f+\mathbf{A}+\mathbf{S}^{0,3}_{k,m})+\mathcal{R}^{0,4}_{k,m}(\varrho,R).$$

The first term in the rhs can be expanded for $\varrho=\Pi(f)$ as

$$\mathcal{R}^{1,2}_{k',m'+2}(\varrho,f+(X^{st}_\chi)_f+\mathbf{A})+\mathcal{R}^{1,4}_{k,m}(\varrho,R).$$

We have for $\varrho=\Pi(f)$

$$\mathcal{R}^{1,2}_{k',m'+2}(\varrho,f+(X^{st}_\chi)_f+\mathbf{A})=\mathfrak{B}_2(\varrho)(f+(X^{st}_\chi)_f+\mathbf{A})^2+\mathcal{R}^{1,3}_{k,m}(\varrho,R),$$

with $\mathfrak{B}_2(\varrho)$ a $C^{m'}$ function with values in $B^2(\Sigma_{-k'},\Sigma_{k'})$ with $\mathfrak{B}_2(0)=0$. Considering the binomial expansion we get for $\varrho=\Pi(f)$

$$\mathcal{R}^{1,2}_{k',m'+2}(\Pi(f'),f')=\mathfrak{B}_2(\varrho)f^2+\widetilde{K}'''+\underline{\mathbf{R}}'''+\mathcal{R}^{0,3}_{k,m}(\varrho,R),$$

with $\underline{\mathbf{R}}'''$ like $\underline{\mathbf{R}}$ and with $\widetilde{K}'''$ like (6.7) with $\widetilde{K}'''(0,b,B)=0$ and $(\widehat{k},\widehat{m})=(k',m')$.
We now set $K=\mathbf{R}^{(1)}_{-1}$ and with the A of (6.47) we write

$$\widetilde{K}'+\widetilde{K}''+\widetilde{K}'''+A=Z''+\widetilde{K}, \tag{6.55}$$

where in Z'' we collect the null terms of the lhs and in $\widetilde{K}$ the other terms. Now we have $K(0)=0$, $\widetilde{K}(0,0,0)=0$ and $\nabla_{b,B}\widetilde{K}(0,0,0)=0$. By Lemma 6.3 for $(\widehat{k},\widehat{m})=(k',m')$ we can choose χ such that for we have

$$\{H_2^{(\ell)},\chi\}^{\widetilde{st}}+Z''+K+\widetilde{K}=0. \tag{6.56}$$

Then $H^{(2)}:=H^{(1)}\circ\phi$ satisfies the conclusions of Theorem 6.4 for $\ell=2$.

$\square$

Summing up, we have proved the following result, whose proof we sketch now.

Theorem 6.5. *For fixed $p_0\in\mathcal{O}$ and for sufficiently large $l\in\mathbb{N}$, there are a fixed $k\in\mathbb{N}$, an $\epsilon>0$, an $1\ll s'\ll l$ and a $1\ll k\ll k'$ such that for solutions $\widehat{U}(t)$ to (2.3) with $\Pi(U)=p_0$ with $|\Pi(\widehat{R}(t))|+\|\widehat{R}(t)\|_{\Sigma_{-k}}<\epsilon$ and $\widehat{R}(t)\in\Sigma_l$, there exists a C^0 map $\Phi:\mathcal{U}^l_{\epsilon,k}\to\mathcal{U}^{s'}_{\epsilon',k'}$ such that*

$$R:=\Phi_R(\Pi(\widehat{R}),\widehat{R})=e^{Jq(\Pi(\widehat{R}),\widehat{R})\cdot\Diamond}(\widehat{R}+\boldsymbol{S}(\Pi(\widehat{R}),\widehat{R})), \tag{6.57}$$

$$\begin{aligned} &\text{with } \boldsymbol{S} \in C^2((-2,2) \times B_{\mathbb{R}^{n_0}} \times B_{\Sigma_{-k}}, \Sigma_{s'}) \\ &\quad q \in C^2((-2,2) \times B_{\mathbb{R}^{n_0}} \times B_{\Sigma_{-k}}, \mathbb{R}^{n_0}) \end{aligned} \tag{6.58}$$

such that $\|\boldsymbol{S}(\Pi(\widehat{R}), \widehat{R})\|_{\Sigma_{s'}} \le C\epsilon \|\widehat{R}\|_{\Sigma_{-k}}$ *and such that splitting* $R(t)$ *in spectral coordinates* $(z(t), f(t))$ *the latter satisfy*

$$\dot{z}_j = \mathrm{i}\partial_{\overline{z}_j} H \ , \quad \dot{f} = J\nabla_f H \tag{6.59}$$

where H *is a given function satisfying the properties of* $H^{(2\boldsymbol{N}+1)}$ *in Theorem 6.4.*

Proof. Since in Lemma 3.7 we can pick arbitrary n, we see by the proof of Theorem 6.4 that we can suppose that the $2\mathbf{N}+1$ transformations ϕ_ℓ are defined by flows (3.18) with pair (r, M) with r and M as large as needed.
Starting with an appropriate $\mathcal{U}^s_{\varepsilon_0,\kappa_0}$, we know that there is a map $\mathfrak{F} : \mathcal{U}^{s'}_{\varepsilon_1,\kappa'} \to \mathcal{U}^s_{\varepsilon_0,\kappa_0}$ as regular as needed which satisfies the conclusions of Theorem 6.4. In particular here we have $s' \gg s$ and $1 \ll \kappa' \ll \kappa_0$ and in $\mathcal{U}^{s'}_{\varepsilon_1,\kappa'}$ we get the system (6.59) by pulling back the system which exists in $\mathcal{U}^s_{\varepsilon_0,\kappa_0}$.
We choose now $l \gg s'$, $1 \ll k \ll \kappa'$ and sufficiently small ϵ and δ with $\mathcal{U}^l_{\delta,k} \subset \mathcal{U}^s_{\varepsilon_0,\kappa_0}$ and $\mathcal{U}^l_{\epsilon,k} \subset \mathcal{U}^{s'}_{\varepsilon_1,\kappa'}$. Here l and κ' can be as large as we want, thanks to our freedom to choose (r, M).
By choosing δ small we can assume $\mathcal{U}^l_{\delta,k} \subset \mathfrak{F}(\mathcal{U}^{s'}_{\varepsilon_1,\kappa'})$. This follows from (3.26) which implies $\mathfrak{F}^{-1}(\mathcal{U}^l_{\delta,k}) \subset \mathcal{U}^l_{\epsilon,k}$. Finally we set $\Phi = \mathfrak{F}^{-1}$ where $\mathfrak{F}^{-1} : \mathcal{U}^l_{\delta,k} \to \mathcal{U}^{s'}_{\varepsilon_1,\kappa'}$.
Formula (6.57) and the information on $\mathbf{S}$ has been proved in the course of the proof of Lemma 4.1. The information on the phase function q can be proved by a similar induction argument, which we skip here.

□

Remark 6.6. The paper [2] highlights in the Introduction and states in Theorem 2.2, that it is able to treat all solutions of the NLS near ground states in H^1. But in fact, in [2] there is no explicit proof of this. While [2] does not state the regularity properties of the maps in Theorem 3.21 and Theorem 5.2 [2], from the context they appear to be just continuous. Even if we assume that they are *almost smooth* transformations (but see Remark 2.10 above), nonetheless an explanation is required on why they preserve the structure needed to make sense of the NLS. But while pullbacks of the Hamiltonian are analyzed, the pullbacks of differential forms and the making sense of them, are not discussed in [2]. For example, there is no explicit discussion on why $\mathfrak{F}^{t*}\Omega_t$ makes sense in formula (3.42) [2], i.e. (3.42) here.

Remark 6.7. In the 2nd version of [2] there is an incorrect effective Hamiltonian. If we use the correct definition of the symbols $\mathbf{S}^{i,j}$ which we give above, the functions $\Phi_{\mu\nu}$ used in the normal form expansion in [2] are in $\mathcal{W}^j$ for some large j, rather than in $\cap_{j\ge 0}\mathcal{W}^j$. In pp. 25–27 in the 2nd version of [2], the $\mathcal{W}^j$'s are defined using the classical pair of operators $L_\pm$, see [13], and are closed subspaces of $H^{j-1}(\mathbb{R}^3)$ of finite codimension. This last fact seems to be unnoticed in [2]

and leads to the breakdown of the proof in the 2nd version of [2], as we explain below. The space $\mathcal{W}^2$, for example, is defined by first considering $\langle L_+u, u\rangle$ for $u \in \ker^\perp L_- \cap \ker^\perp L_+ \subset L^2$. Notice that $\langle L_+u, u\rangle \geq 0$, see Prop. 2.7 [13] or Lemma 11.12 [10]. Proceeding like in Lemma 11.13 [10] it can be shown that for $u \in \ker^\perp L_- \cap \ker^\perp L_+ \subset L^2$ with $u \neq 0$ we have $\|u\|_L^2 := \langle L_+u, u\rangle > 0$. Then consider the completion of $\ker^\perp L_- \cap \ker^\perp L_+ \cap C_0^\infty$ by the norm $\|u\|_L$. This completion is exactly $\ker^\perp L_- \cap \ker^\perp L_+ \cap H^1(\mathbb{R}^3)$. Then $\mathcal{W}^2$ is a closed subspace of finite codimension of the latter space. Specifically, $\mathcal{W}^2$ is in the continuous spectrum part in the spectral decomposition of the operator L_-L_+, which is selfadjoint for $\langle u, v\rangle_L := \langle L_-^{-1}u, v\rangle$ in $\ker^\perp L_-$. Notice that, under hypotheses analogous to (L1)–(L6) in Sect. 5, L_-L_+ has finitely many eigenvalues and its eigenfunctions are Schwartz functions. Likewise, also the other $\mathcal{W}^j$'s are closed subspaces of $H^{j-1}(\mathbb{R}^3)$ of finite codimension. Later in the 2nd version of [2], at p.41, the Strichartz estimates hinge on the false inclusion of $\mathcal{W}^j$, or of $\mathcal{W}^\infty$, in $L^{\frac{6}{5}}(\mathbb{R}^3, \mathbb{C})$. Additional mistakes appear in the justification of the Fermi Golden rule. While formulas $R^{\pm}_{L_0}(\rho)\Phi$ in (St.2)–(St.3) on p. 38 of the 2nd version make sense because $\Phi \in H^{k,s}$ for $s > 0$ appropriate, analogous formulas $R^{\pm}_B(\rho)\Phi$ in (6.50) and elsewhere in Sect. 6.2, are undefined when we know only that $\Phi \in \mathcal{W}^\infty$. In fact even $R^{\pm}_{-\Delta}(\rho)\Phi$ is undefined for $\rho \geq 0$ for such Φ's. So in particular, in the 2nd version of [2], the discussion of the Fermi Golden rule is purely formal.

The above ones are not simple oversights. Rather, they stem from the fact that, in the 2nd version of [2], the homological equations are solved only in these $\mathcal{W}^j$'s, while it is unclear if they can be solved in spaces with spacial weights like the $H^{k,n}$ or the Σ_n for $n > 0$, as we remarked in an early version of [6]. The 3rd version of [2] credits our remark for having stimulated changes in this part of the paper. These changes are classified in the 3rd version of [2] as plain simplifications. This might leave the wrong impression that the proof in the 2nd version of [2], although more complicated than in the 3rd version, is still correct.

7 The NLS and the Nonlinear Dirac Equation

We give a sketchy discussion of few examples.

The Nonlinear Schrödinger equation. We consider the equation

$$\mathrm{i}U_t = -\Delta U + 2B'(|U|^2)U \ .$$

Here $N = 1$, $\mathcal{D} = -\Delta$, $|\ |_1 = |\ |$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. There are four invariants:

$$Q(U) = \Pi_4(U) = \frac{1}{2}\langle U, U\rangle \text{ and } \Pi_j(U) = \frac{1}{2}\langle U, J\frac{\partial}{\partial x_j}U\rangle \text{ for } j \leq 3.$$

For fixed $v \in \mathbb{R}^3$ we have

$$Q(e^{-\frac{1}{2}Jv\cdot x}U) = Q(U)\ ,\ \Pi_j(e^{-\frac{1}{2}Jv\cdot x}U) = \Pi_j(U) - \frac{v_j}{2}Q(U) \text{ for } j \le 3 \text{ and}$$

$$E(e^{-\frac{1}{2}Jv\cdot x}U) = E(U) - \sum_{j=1}^{3} v_j \Pi_j(U) + \frac{v^2}{2}Q(U).$$

There is well established theory guaranteeing under appropriate hypotheses existence of open sets $\mathcal{O} \subseteq \mathbb{R}^+$ and $(\phi_\omega, 0) \in C^\infty(\mathcal{O}, \mathcal{S}(\mathbb{R}^3, \mathbb{R}^2))$ such that

$$\Delta\phi_\omega - \omega\phi_\omega + 2B'(\phi_\omega^2)\phi_\omega = 0 \quad \text{for } x \in \mathbb{R}^3.$$

More precisely it is possible to prove exponential decay to 0 of $\phi_\omega(x)$ as $x \to \infty$. For $v \in \mathbb{R}^3$ arbitrary we get $\Phi_p(x) = e^{-\frac{1}{2}Jv\cdot x}(\phi_\omega(x), 0)$ where $p_4 = \Pi_4(\phi_\omega)$ and $p_j = -\frac{1}{2}v_j p_4$ for $j \le 3$. We have $\lambda_4(p) = -\omega - \frac{v^2}{4}$ and $\lambda_j(p) = -v_j$ for $j \le 3$. Notice that for $\frac{d}{d\omega}Q(\phi_\omega) \neq 0$ this yields (2.7). Notice that

$$\nabla^2 E(e^{-\frac{1}{2}Jv\cdot x}U) = e^{-\frac{1}{2}Jv\cdot x}\left(\nabla^2 E(U) - Jv\cdot\nabla_x + \frac{v^2}{4}\right)e^{\frac{1}{2}Jv\cdot x}$$

and that $v\cdot\nabla_x \circ e^{-\frac{1}{2}Jv\cdot x} = e^{-\frac{1}{2}Jv\cdot x} \circ (v\cdot\nabla_x - J\frac{v^2}{2})$ and

$$\nabla^2 E(\Phi_p(x)) - \lambda(p)\cdot\Diamond = e^{-\frac{1}{2}Jv\cdot x}\left(\nabla^2 E((\phi_\omega, 0)) - Jv\cdot\nabla_x + \frac{v^2}{4}\right)e^{\frac{1}{2}Jv\cdot x}$$
$$+ Jv\cdot\nabla_x e^{-\frac{1}{2}Jv\cdot x}e^{\frac{1}{2}Jv\cdot x} + (\omega + \frac{v^2}{4})e^{-\frac{1}{2}Jv\cdot x}e^{\frac{1}{2}Jv\cdot x}.$$

They imply

$$\mathcal{H}_p = e^{-\frac{1}{2}Jv\cdot x}\mathcal{H}_\omega e^{\frac{1}{2}Jv\cdot x}\ , \quad \mathcal{H}_\omega := J(\nabla^2 E((\phi_\omega, 0)) + \omega). \tag{7.1}$$

The multiplier operator $e^{-\frac{1}{2}Jv\cdot x}$ is an isomorphism in all spaces Σ_n so all the information on the spectrum of $\mathcal{H}_p$ is obtained from the spectrum of $\mathcal{H}_\omega$. We have $\mathcal{H}_\omega = \mathcal{H}_{0\omega} + V$ where $H_{0\omega} := J(-\Delta + \omega)$ and

$$V := 4J\begin{pmatrix} -B'(\phi_\omega^2) - 2B''(\phi_\omega^2)\phi_\omega^2 & 0 \\ 0 & -B'(\phi_\omega^2) \end{pmatrix}.$$

This yields $\sigma_e(\mathcal{H}_\omega) = \sigma(H_{0\omega}) = (-\infty, -\omega] \cup [\omega, \infty)$ and that $\sigma_p(\mathcal{H}_\omega)$ is finite with finite multiplicities. The fact that $\sigma_p(\mathcal{H}_\omega)$ is in the complement of $\sigma_e(\mathcal{H}_\omega)$ is expected to be true generically. Set $\mathcal{H} = \mathcal{H}_\omega P_c(\omega)$ for $P_c(\omega)$ the projection on $X_c(\mathcal{H}_\omega)$.

Lemma 7.1. *The statement in (A5) is true.*

Proof. Notice that Σ_n is invariant by Fourier transform so that (2.4) is equivalent to the fact that for the following multiplier operator (that is an operator $\psi(x)$ which maps $u \to (\psi u)(x) := \psi(x)u(x)$) we have

$$\|(1+\epsilon^2+\epsilon^2|x|^2)^{-2}\|_{B(\Sigma_n,\Sigma_n)} \le C_n < \infty \ \forall \ |\epsilon| \le 1 \text{ and } n \in \mathbb{N}. \tag{7.2}$$

Similarly (2.5) is equivalent to

$$\begin{aligned} &strong - \lim_{\epsilon\to 0}(1+\epsilon^2+\epsilon^2|x|^2)^{-2} = 1 \text{ in } B(\Sigma_n,\Sigma_n) \\ &\lim_{\epsilon\to 0}\|(1+\epsilon^2+\epsilon^2|x|^2)^{-2}-1\|_{B(\Sigma_n,\Sigma_{n'})} = 0 \quad \text{for any } n' \in \mathbb{N} \text{ with } n' < n. \end{aligned} \tag{7.3}$$

Both (7.2)–(7.3) are elementary to check using the first definition of Σ_n in Sect.2, computing commutators of the multiplier operators with ∂_x^α and computing elementary bounds on the derivatives of the multipliers. □

Lemma 7.2. *The statement in (A6) is true.*

Proof. Using the Fourier transformation like in Lemma 7.1, (A6) is equivalent to the statement that for any $n \in \mathbb{N}$ and $c > 0$ there a C s.t. the following multiplier operator satisfies

$$\|e^{(1+\epsilon^2+\epsilon^2|x|^2)^{-2}J(\tau_4-\sum_{j=1}^3 x_j\tau_j)}\|_{B(\Sigma_n,\Sigma_n)} \le C$$

for any $|\tau| \le c$ and any $|\epsilon| \le 1$. This too is elementary to check.

Lemma 7.3. *The statement in (L7) is true.*

Proof. From $\sigma(\mathcal{H}) = \sigma_e(\mathcal{H}_\omega)$ we have $R_{\mathcal{H}} \in C^\omega(\rho(\mathcal{H}), B(L^2,L^2))$.
We have $R_{\mathcal{H}_{0\omega}}$ and $R_{\mathcal{H}_{0\omega}}\partial_{x_j}$ are in $C^\omega(\rho(\mathcal{H}), B(\Sigma_n,\Sigma_n))$ for any $n \in \mathbb{N}$. By conjugation by Fourier transform this is equivalent to the statement that for $z \in \rho(\mathcal{H}_{0\omega})$ and $i = 0,1$, we have

$$\xi_j^i \begin{pmatrix} (|\xi|^2+\omega-z)^{-1} & 0 \\ 0 & -(|\xi|^2+\omega+z)^{-1} \end{pmatrix} \in B(\Sigma_n,\Sigma_n).$$

This is elementary, using the first definition of Σ_n in Sect.2.
We have for $i = 0,1$

$$R_{\mathcal{H}}(z)\partial_{x_j}^i = R_{\mathcal{H}_{0\omega}}(z)P_c(\omega)\partial_{x_j}^i - R_{\mathcal{H}_{0\omega}}(z)VR_{\mathcal{H}}(z)\partial_{x_j}^i. \tag{7.4}$$

From (7.4) we derive, for $\| \ \| = \| \ \|_{B(L^2,L^2)}$.

$$\|R_{\mathcal{H}}(z)\partial_{x_j}^i\| \le \|(1+R_{\mathcal{H}_{0\omega}}(z)V)^{-1}\|\|R_{\mathcal{H}_{0\omega}}(z)P_c(\omega)\partial_{x_j}^i\|, \tag{7.5}$$

which yields the $n = 0$ case.
From (7.4) we derive

$$\begin{aligned} &\|R_{\mathcal{H}}(z)\partial_{x_j}^i\|_{B(\Sigma_n,\Sigma_n)} \le C\|R_{\mathcal{H}_{0\omega}}(z)\partial_{x_j}^i\|_{B(\Sigma_n,\Sigma_n)} \\ &+ C\|R_{\mathcal{H}_{0\omega}}(z)\|_{B(\Sigma_n,\Sigma_n)}\|\langle x\rangle^n V\|_{W^{n,\infty}}\|R_{\mathcal{H}}(z)\partial_{x_j}^i\|_{B(H^n,H^n)}. \end{aligned}$$

The last factor is bounded. Indeed for $\mathbf{v} = R_{\mathcal{H}}(z)\partial^i_{x_j}\mathbf{u}$ we have

$$\partial^\alpha_x \mathbf{v} = R_{\mathcal{H}}(z)\partial^\alpha_x \partial^i_{x_j}\mathbf{u} + R_{\mathcal{H}}(z)[V, \partial^\alpha_x]\partial^i_{x_j}\mathbf{u}$$

and induction in n yields the desired bounds $\|\mathbf{v}\|_{H^n} \le C\|\mathbf{u}\|_{H^n}$. $\square$

The Nonlinear Dirac Equation. Here the unknown U is $\mathbb{C}^4$-valued, u^* its complex conjugate and for $m > 0$

$$\mathrm{i}U_t - D_m U - Vu + 2B'(U \cdot \beta U^*)\beta U = 0 \tag{7.6}$$

where we assume for the moment $V = 0$ and where $D_m = -\mathrm{i}\sum_{j=1}^3 \alpha_j \partial_{x_j} + m\beta$, with for $j = 1, 2, 3$

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix},$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notice that the symmetry group (7.6) is not Abelian. In [4] there is a symmetry restriction on the solutions considered, by looking only at functions such that for any $x \in \mathbb{R}^3$ we have $U(-x) = \beta U(x)$ and $U(-x_1, -x_2, x_3) = S_3 U(x_1, x_2, x_3)$ with $S_3 := \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$. We need to redefine the spaces Σ_n in the proof, introducing these symmetries. This does not affect the proof.
There is a unique invariant $Q(U) = \frac{1}{2}\|u\|_{L^2}$. In this case $\Diamond_1 U = U$ for any u. Hence all the changes of variables are diffeomorphism within each space $\mathcal{P}^K$ (or $\widetilde{\mathcal{P}}^K$).
(A5)–(A6) in this case are elementary. In fact (A5) is unnecessary, (A6) is necessary only for $\epsilon = 0$, in which case is trivial. (L7) is necessary only for $i = 0$ (given that the only $\Diamond_j$ is the identity) and can be proved in a way similar to Lemma 7.3.

Nonlinear Dirac Equation with a Potential. Pick $V \in \mathcal{S}(\mathbb{R}^3, B(\mathbb{C}^4))$ with $V(x)$ selfadjoint for the scalar product in $\mathbb{C}^4$ for any $x \in \mathbb{R}^3$. Then generically $\sigma_p(D_m + V) \subset (-m, m)$. Suppose $\sigma_p(D_m + V) = \{e_0, ..., e_{\mathbf{n}}\}$ with $e_0 < ... < e_{\mathbf{n}}$. Then bifurcation yields corresponding families of small standing waves $e^{-\mathrm{i}\omega t}\phi_\omega(x)$ of (7.6). For generic V the e_j have multiplicity 1. If we focus on e_0, for generic smooth $B'(r)$ there will be a smooth family $\omega \to \phi_\omega$ in $C^\infty(\mathcal{O}, \Sigma_n)$ for any n, with $\mathcal{O}$ an open interval one of whose endpoints is e_1. Then it can be shown that for generic V the hypotheses (L1)–(L6) in Sect. are true, as well as all the previous hypotheses. Indeed in this case, taking ω sufficiently close to e_0, we have eigenvalues with $\mathbf{e}'_j$ arbitrarily close to $e_j - e_0$. Generically this yields (L4)–(L5). The multiplicity of the $\mathrm{i}\mathbf{e}'_j$ is 1. We have $\sigma_e(\mathcal{H}_\omega) = (-\infty, -m + |\omega|] \cup [m - |\omega|, \infty)$. An eigenvalue λ of $\mathcal{H}_\omega$ is either $\lambda = 0$, or $\lambda = \pm\mathrm{i}\mathbf{e}'_j$ for some j. This in particular yields (L1)–(L3).

References

[1] R.Abraham, J.Marsden, T.Ratiu *Manifolds, Tensor Analysis and Applications*, Springer, 2000.

[2] D.Bambusi, *Asymptotic stability of ground states in some Hamiltonian PDEs with symmetry*, arXiv:1107.5835v3, version of the 24th February 2012.

[3] D.Bambusi, S.Cuccagna, *On dispersion of small energy solutions of the nonlinear Klein Gordon equation with a potential*, Amer. Math. Jour., 133 (2011), 1421–1468 .

[4] N.Boussaid, S.Cuccagna, *On stability of standing waves of nonlinear Dirac equations*, to appear Comm. Part. Diff. Eq..

[5] V.Buslaev, G.Perelman, *On the stability of solitary waves for nonlinear Schrödinger equations*, Nonlinear evolution equations, editor N.N. Uraltseva, Transl. Ser. 2, 164, Amer. Math. Soc., 75–98, Amer. Math. Soc., Providence (1995).

[6] S.Cuccagna, *On asymptotic stability of moving ground states of the nonlinear Schrödinger equation*, to appear Trans. Amer. Math. Soc.

[7] S.Cuccagna, *The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states*, Comm. Math. Physics, 305 (2011), 279-331.

[8] S.Cuccagna, *On instability of excited states of the nonlinear Schrödinger equation*, Physica D, 238 (2009), 38–54.

[9] S. Cuccagna, *On scattering of small energy solutions of non autonomous hamiltonian nonlinear Schrödinger equations*, J. Differential Equations, 250 (2011), 2347-2371.

[10] I.Rodnianski, W.Schlag, A.Soffer, *Asymptotic stability of N-soliton states of NLS* , (2003), arXiv:math/0309114v1.

[11] I.M.Sigal, *Nonlinear wave and Schrödinger equations. I. Instability of periodic and quasi- periodic solutions* , Comm. Math. Phys. 153 (1993), pp. 297–320.

[12] A.Soffer, M.I.Weinstein, *Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations* , Invent. Math., 136 (1999), 9–74.

[13] M.I.Weinstein, *Modulation stability of ground states of nonlinear Schrödinger equations*, Siam J. Math. Anal. 16 (1985), 472–491.

Department of Mathematics and Geosciences, University of Trieste, via Valerio 12/1 Trieste, 34127 Italy

E-mail Address: `scuccagna@units.it`